Collusion

New insights on theory, empirical methods and history

by

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"You have to learn the rules of the game.

And then you have to play better than anyone else."

Albert Einstein

Biographical note

Pedro Cardoso Pereira Silva Gonzaga was born on October the 13^{th}, 1990, in Porto, Portugal.

Pedro Gonzaga graduated in Economics from Faculty of Economics of University of Porto (FEP), in 2011, with the final grade 18 (out of 20). During the course he was awarded with three merit scholarships for outstanding achievement in the academic years 2008/2009, 2009/2010 and 2010/2011, the *Incentive Prize* 2008/2009, the *Talent FEP 2011* award and the first prize in the contest *Applied Research in Economics and Management* in 2011, for the paper “Effect of the minimum wage on employment and welfare”. In the second half of the course he conducted research work in industrial economics and accessing price and spent the summer of 2011 working on an internship in Portuguese Commercial Bank (BCP).

After graduating, Pedro was accepted into the doctoral program in Economics at the Faculty of Economics of University of Porto, with a research grant awarded by FCT (The Foundation for Science and Technology). In 2013 he concluded the scholar component of the program with the final grade 18 (out of 20) and by the middle of 2014 he had written the four papers on collusion for his PhD dissertation. Many of these works were presented at FEP PhD Seminars in 2013 and 2014, at the UECE Lisbon Meetings 2013 and in an exclusive conference lectured in the Portuguese Competition Authority in January 2014. One of the works was also accepted in the 41^{st} annual conference of the European Association for Research in Industrial Economics (EARIE 2014), in Milan.

Pedro likes to play the organ and to do ballroom dance, illusionism, computer programming, film editing and strategic games, such as chess and *Catan*, though his greater passion remains industrial economics.

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Abstract

Collusion, as a highly profitable crime with severe consequences to society, has attracted the attention of many economic researchers. While most subjects have already been extensively debated in the collusion literature, we provide new insights on topics that have been rarely addressed and whose solutions are yet to be discovered. This dissertation is composed by a literature review, two theoretical chapters and one empirical work.

The first chapter is a historical analysis of some important contributions to collusion literature. It is our purpose to illustrate how the evolution of game theory has changed our understanding about the strategic interaction between the members of cartels.

The second chapter studies the economic effects of cooperative wage fixing in industries with a single input. We establish a deterministic relation between collusion in the labor market and collusion in the product market and we show that both types of collusion lead to higher prices, lower wages, lower employment and smaller quantities transacted, due to the elimination of the business and labor force stealing effects.

The third chapter extends the previous analysis to industries that use two types of labor. We consider a two-stage game where firms hire non-specialized workers for an exogenous wage and specialized workers whose wages can be cooperatively determined. It is shown that semi-collusion in the labor market has the same qualitative results than complete collusion, although new dynamic effects and strategies are found.

In the forth chapter we develop an empirical method to detect price collusion from the observation of economic data. Our approach describes the supply side of the industry as a switching regression with two regimes, collusion and competition, which is estimated using a modified expectation-maximization algorithm. We use simulated data to show that our algorithm accurately predicts collusion and consistently estimates the parameters of the switching regression.

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1.1. Introduction

The creation of cartels has been for a long time a major concern for economists, due to the severe effects that anticompetitive practices as artificial increases in prices, output restrictions and division of markets have on the efficiency of the economy and on social welfare. Indeed, the phenomenon of collusion was already warned by Adam Smith in 1776, who stated that:

> "People of the same trade seldom meet together, even for merriment and diversion, but the conversation ends in a conspiracy against the public, or in some contrivance to raise prices.[1]"

This concern has raised in most countries the creation of several antitrust laws (as the Sherman Act approved in 1890 in the United States) as well as the creation of competition authorities responsible to prosecute collusive practices. However, the most important contributions to the economic literature on the theory of collusion would only come much later.

Because collusion commonly occurs in oligopolistic industries composed by a small number of large firms, it could not be properly investigated using, for example, the general equilibrium paradigm, where markets are assumed to be perfectly competitive. Instead, it was only after the creation of a theory of games in the mid-twentieth century that it was possible to investigate how the strategic interaction between firms lead to the creation of a cartel and to analyze several other key issues: How do firms coordinate cooperative strategies compatible with rational behavior? Which firms of the industry compose the cartel? How do cartelized firms distribute profits among them? Under what conditions collusion can be sustained? Can firms collude without being able to communicate with each other? What is the role of competition authorities?

In order to answer all these questions, many years of theoretical development were required and, above all, it took a deep evolution of the economic thought about the cooperative nature of collusion. While in the initial state of theory collusion was considered a cooperative game between firms (as our intuition would suggest), it turned out that in some components of a collusive game firms cannot collaborate with each

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[1] Quoted from *An Inquiry into the Nature and Causes of the Wealth of Nations*

other. And the continuous changes in the economic environment and regulation along time caused economists to increasingly study collusion as a non-cooperative game.

The purpose of this chapter is thus to illustrate how has collusion theory evolved from the use of cooperative games to non-cooperative games and how that theoretical/methodological evolution affects our scientific knowledge on the subject. This will be done according to the following structure:

In Section 1.3 we briefly present the theory of games as it was originally developed by von Neumann and Morgenstern in 1943, which focuses mainly on games where coalitions are formed and at least some players cooperate with each other. We discuss the main contributions of this theory to the study of cartels and, withal, how it fails to solve some essential issues.

In Section 1.4 we study the context in which the concept of Nash equilibrium or non-cooperative equilibrium was introduced and, in particular, we show how this concept was applied to cooperative games to overcome some imperfections of the last theory. At this point, while collusion is still studied by economic science as a cooperative game, the economic literature uses the Nash equilibrium to solve the conflicting interests that arise at some stages of the game in which firms are not able to cooperate with each other.

In Section 1.5 we discuss the ongoing changes in the economic environment that caused firms to increasingly coordinate their collusive strategies without talking or meeting (the so called informal cartels). This led some economists to investigate the conditions under which collusion can be sustained without communication or enforceable agreements, initiating thereby the research of non-cooperative collusion that currently dominates economic literature.

For now, in Section 1.2, we define some important concepts that we have already used so far, but whose precise meaning is crucial for a rigorous discussion about the historical evolution of collusion theory.

1.2. Terminology

The first concept to be defined is naturally our object of study. In the economic sense, collusion consists in any explicit or tacit agreement between firms to raise prices, restrain the output produced, divide markets or follow any other anti-competitive practice that increases the profits of the industry at the expense of social welfare. This economic definition embraces both formal cartels where firms actually meet and undertake with each other, as well as informal cartels in which firms can only communicate through the mechanisms of the market.

In addition to our central concept we must provide some notions in game theory. Briefly, a game is a set of rules that defines what moves or actions can be executed by each player at each stage. The rules of the game also associate any sequence of moves with a payoff or outcome per player. In particular we are interested in the notions of cooperative and non-cooperative game, which differ by the fact that in the former players are free to communicate, make enforceable agreements and to do side payments outside the rules of the game.

During the course of a game it is common to define a strategy, which consists in a set of rules describing how a particular player should behave in every possible situation that might arise. Notice that in opposition to the unbreakable rules of the game that define the legal and illegal actions, strategies are only guides that suggest the execution of a specific action among all possible actions.

At last, the solution of the game is the set of strategies that players are likely to choose when they behave rationally. Sometimes the term solution is used to express the distribution of payoffs that results from all players taking their optimal strategies. This is the concept of solution as a set of imputations defined by von Neumann and Morgenstern (1943).

We have now the necessary tools to start our journey. Any other technical concepts that may be used throughout Chapter 1 will be defined at the appropriate time.

1.3. The contribution of cooperative games to collusion theory

Modern game theory is based on one of the most important mathematical works ever, the *Theory of Games and Economic Behavior*, written by John von Neumann and Oskar Morgenstern and published for the first time in 1944 by the Princeton University Press. This book, which was created after a theory developed and published in 1928 by the former author, has introduced the tools of modern logic in economics, according to the Journal of Political Economics, and was admired by “the audacity of vision, the perseverance in details, and the depth of thought displayed in almost every page” by the American Economic Review.

It is not clear why John von Neumann, a brilliant mathematician with so many important contributions to mathematics, physics and computer sciences, was concerned in creating a theory of games applied to economic and social problems. Perhaps his motivation can partially be explained from the fact that, having born on 1903 in Budapest, he lived his youth in wartimes and in contact with several games specifically designed for military schools (like *Kriegspiel*), where the prediction of the moves and strategies of the other players is extremely important. In addition to his contact with strategic war games, according to William Poundstone “The nominal inspiration for game theory was poker, a game von Neumann played occasionally and not especially well”. Indeed, unlike other luck games as blackjack and roulette where probability theory alone can be used to optimize gains, poker players must account not only for the odds but also for the information they reveal with certain actions (which they can avoid by bluffing). A poker player who always bets high when he has a good hand and bets low otherwise is quickly found out by his opponents and can easily be beaten.

The fact that Von Neumann foresaw a formal theory with wide applications just from the observation of games where the actions of players depend on one another required an enormous talent. And his premature genius becomes evident taking into account that he received advanced instruction in mathematics from private tutors during his childhood, he got a PhD in mathematics with only 22 years at the Pázmány Péter University in Budapest and, in 1930, he was one of the first four people being selected to the Institute for Advanced Study in Princeton, along with Albert Einstein.

The application of theory of games to economic problems and the use of economic concepts as utility, rational behavior and perfect/ imperfect information were largely due to the collaboration of Oskar Morgenstern, an economist born in German on 1902, graduated in the University of Vienna in 1925 and doctorate in political science. After graduating, Morgenstern became interested on the analyses of what he called "live variables", that is, variables that depend on the economic actions and decisions of other agents. Among several studies, he investigated the effects of predictions on the predicted events and created an example where he showed that the intellectual battle between Sherlock Holmes and Professor Moriarty could not be solved with an infinite reasoning of the type "Sherlock Holmes predicts that Professor Moriarty predicts that Sherlock Holmes predicts…" and, instead, it could only be solved with an arbitrary decision. It was during the presentation of these problems in the Karl Menger's colloquium that Morgenstern was approached by the mathematician Edward Čech, who remarked the strong connection between his ideas and those presented on a paper published in 1928 by John von Neumann. And, when Morgenstern came to the United States in 1938, one of the main reasons why he accepted so quickly the invitation from Princeton University to lecture political economy was precisely his desire to meet and interact with von Neumann. It didn't take long until the economist and the mathematician initiated the joint creation of one of the most important works of their life, *The Theory of Games and Economic Behavior*.

The theory presented by von Neumann and Morgenstern focus in the resolution of cooperative games, where coalitions are typically formed by players. Although the authors do not explicit clarify why they are particularly interested in cooperative games, we can conjecture by putting ourselves in their position. Anyone who shall create a new and innovative theory from scratch should attempt to do it as general as possible and the authors had absolutely no reason to suppose that, in most relevant games, players could not freely communicate nor sign enforceable contracts, given the society they lived based on freedom and private property. Besides, because players can always improve or maintain their payoffs by cooperating with each other (in the worst case players choose the same strategies they would if they were not able to cooperate), it would be completely irrational not to attempt to form coalitions. Finally, their theoretical analysis of cooperative games may simply have resulted from the evidence they observed in some important strategic “games” like wars. In World War II we do not observe numerous countries fighting each other in a complete anarchy, but the formation of two big coalitions: the allies and the axis powers.

We have shown the historical conditions that led to the creation of an authentic masterpiece on the theory of games and, specifically, on the theory of cooperative games. Now we must question how this theory can be used to find the solution of a game and to increase our knowledge about collusion.

When using the term solution, von Neumann and Morgenstern usually mean a set of imputations, that is, the set of all possible distributions of payoffs that are likely to occur in a specific game. In order to find the set of imputations that correspond to the solution of the game, the authors apply the principle of dominance. Despite not having the time to explain here the mathematical concept of domination, we can provide a simple intuitive notion: we say that an imputation x dominates an imputation y if there are a sufficient number of players strictly preferring x to y and if those players are able to enforce the imputation x to the other players of the game.

If the concept of domination was transitive, the solution of the game would become trivial and it would simply correspond to the set of imputations that dominate all the others. Unfortunately that is not the case, as it is possible that an imputation x dominates an imputation y, which dominates an imputation z, which in turn dominates the initial imputation x. Although the intransitive characteristic of domination makes the job of finding a solution harder, it surely explains the cyclical behavior of many social organizations, where the solution changes according to the specific lobby that is enforcing their preferences at a given moment of time.

Using the intransitive concept of domination, von Neumann and Morgenstern proved that the solution of a game corresponds to a set of imputations S that verify two distinct properties: firstly, no imputation contained in S can be dominated by another imputation contained in S; and secondly, any imputation not contained in S must be dominated by at least one of the imputations in S. This result was probably one of the most important accomplishments of their work and it was essential to the subsequent evolution of game theory.

With respect to the theory of collusion, the concept of solution based on the principle of domination allowed us to solve many previously unsolved problems. In fact, given an industry composed by a limited number of profit-optimizing firms, we are now able to find what coalitions (or cartels) may be formed in the industry, the profits earned by each coalition and the strategies that must be played to obtain such profits (that is, the prices, quantities produced or any other control variable that firms must set). Notwithstanding, their concept of solution was still not enough to resolve other important issues, as one should expect from any newborn theory. We will illustrate some of those failures using examples of specific games.

The simplest game presented in the von Neumann and Morgenstern's book is the zero-sum two-person game, which is composed by two players whose sum of payoffs is always equal to zero. Since in this game the gain of a player is the loss of the other, no coalitions can be formed and so this is not an interesting example from the standpoint of collusion theory.

The second simplest game is the zero-sum three-person game. In this case three different coalitions can be formed (player 1 plus player 2, player 2 plus player 3 and player 3 plus player 1) and the solution of the game correspond to the three imputations of payoffs that result from the three possible alliances. Although in practice we only observe one coalition being formed, the other two coalitions are always present in an implicit way, as they affect the distribution of payoffs between the two allied players.

Suppose, for instance, that in a particular zero-sum three-person game, the coalition formed by players 1 and 2 is able to get the payoff c (while player 3 earns $-c$), the coalition formed by players 1 and 3 is able to obtain b (while player 2 receives $-b$) and finally the coalition of players 2 and 3 is able to extract the payoff a from player 1. Suppose further that player 1 demands an amount x to anyone who desires to ally with him. If x is such that the sum of the payoffs that player 2 and player 3 are able to collect when each of them cooperates with player 1 is lower than the payoff player 2 and player 3 can get when the two form a coalition, no one will be willing to cooperate with player 1. In other words player 1 must demand an amount x such that $(c-x)+(b-x) \geq a \leftrightarrow x \leq \frac{-a+b+c}{2}$ if he wants to join any coalition. Similarly, player 2 must demand an amount $y \leq \frac{a-b+c}{2}$ and player 3 must demand an amount $z \leq \frac{a+b-c}{2}$. This means that each player will require to his potential ally the maximum amount he can possibly demand. He cannot ask for more due to the implicit threat of the other two players forming a coalition themselves.

It follows that all players wish to belong to the only coalition of the game, but they don't care to whom they colligate, since they always receive the same share of the payoffs as long as they belong to the alliance. Unfortunately this raises a remarkable problem: we have absolutely no way to determine what coalition is going to be formed, once every player is completely indifferent between cooperating with any of the other two players.

This is one of the previously mentioned limitations of applying the principle of dominance to collusion theory. In many situations we cannot use the solution proposed by von Neumann and Morgenstern to determine which firms of the industry will form the cartel, for the simple reason that no particular coalition dominates the other.

Another limitation of this concept of solution can be illustrated in a general non-zero-sum two-person game. Remark that in this type of game a coalition is likely to occur, once the two players can joint their forces for mutual benefit.

In the *Theory of Games and Economic Behavior* all general non-zero-sum n-person games are reduced to zero-sum games with $n + 1$ players and solved afterwards. This is done through the creation of an artificial player that has no interaction on the game but who earns the symmetric of the payoffs of the other players, so that the sum of the payoffs of the $n + 1$ players is null. In our case the non-zero-sum two-person game can thereby be transformed in a zero-sum game with three players.

However, this game is different from the last example we gave. Because the third player is artificial the only possible coalition is between the two real players and there is no more an imminent threat of one of the players colligating with the third artificial player. Unfortunately, this also means that there is no way of determining how the two cooperating players will distribute the payoffs between them. As long as each of the players is better off when cooperating with the other, any distribution is possible.

Once again, a major problem faced by a cartel is how to distribute the total profits among its members. And the principle of domination fails again to provide an answer, because no specific repartition of profits dominates the other.

It seems that the concept of dominance suggested by von Neumann and Morgenstern embraces too many solutions. Fortunately it didn’t take long until a more selective concept was created by the next mastermind in game theory, John Nash.

1.4. Applying the Nash equilibrium to cooperative collusion

John Forbes Nash Jr. (born on 1928 in West Virginia) is one of the most important and worldwide famous mathematicians, whose contributions in game theory were essential to the development of various fields in economics, including collusion. Nash has been acknowledge for his genius mind, quick thinking, long lasting memory and specially for his impressive mathematical intuition, which allowed him to foresee the solution of the most complex problems that other brilliant mathematicians were not able to solve. In the words of the mathematician Donald Newman:

> "Everyone else would climb a peak by looking for a path somewhere on the mountain. Nash would climb another mountain altogether and from that distant peak would shine a searchlight back onto the first peak."

Nash has also been known for other peculiar characteristics of his personality. His arrogance and contempt for the established knowledge led him to attempt to discredit and correct Albert Einstein's theory of relativity, even though he had never studied physics before. Due to his compulsive rationality he used to seek mathematical solutions for daily problems, as whether to take the elevator or to use the stairs. And his schizophrenic illness led to very strange events, as Nash sending letters to American embassies claiming that all men who wore red ties were conspiring to form a communist government.

Between 1945 and 1948 Nash attained the Carnegie Institute of Technology, where he initially entered as a chemical engineering student, changing later to chemistry and once again to mathematics. At the end of the course he was not only awarded with the Bachelor degree, but also with a Master of Science, due to the high progress he had as a student. At the Carnegie Institute he also took a course in International Economics, which contributed to the ideas behind the paper we wrote on *The Bargaining Problem*, which in turn raised his interest in game theory.

During this period Nash entered a few times the William Lowell Putnam Mathematical Competition, a renowned mathematical contest that rewarded the top 5 winners with a nominal prize and recognition. The contest involved solving twelve complex problems within a six hours time limit, whose difficulty was such that more than half of the competitors were not able to score a single point. Although Nash made it to the top 10 in his second attempt, he didn't get to the top 5, a hard blow that he took pretty badly. However, this failure turned out crucial to the history of game theory.

After taking his Master degree, Nash was accepted into the four most prestigious universities in mathematics: Harvard, Princeton, Chicago and Michigan. Nash had been attracted for long by the reputation and status of Harvard, which was initially his first choice. However the fellowship offered by Princeton was slightly better, in part because Harvard had always valued too much the results obtained in the Putnam competition, while the second university didn't care about the test score. Thus, when Princeton finally offered him the most prestigious and important fellowship, the John S. Kennedy fellowship, Nash accepted it, as he considered that Harvard did not valued him enough. And it was precisely at Princeton – where von Neumann and Morgenstern were at that time – that John Nash produced his 27 pages PhD thesis containing the probably most important contribution of his life, the non-cooperative equilibrium or, as it would be denominated later, the Nash equilibrium.

The concept of non-cooperative equilibrium reflects a stable situation in general n-person games where players act rationally, independently and on behalf of their self interest only, without collaborating with each other. At the Nash equilibrium each player chooses the strategy that maximizes his payoff when the strategies of the other players are hold fixed or, in other words, at the Nash equilibrium there are no profitable deviations, since no agent can improve his gains as long as the other players keep their strategies constant. Nash (1951) proved the existence of a non-cooperative equilibrium in any general n-person game by using a fixed point theorem, though it may not be unique. The non-cooperative equilibrium can be used to find the solution of a non-cooperative game or, at least, its sub-solutions.

There is a clear difference between the methods used by von Neumann and Nash to solve n-person games. While von Neumann considered a communicative and cooperative behavior between the players of a game, leading to the formation of groups or coalitions, Nash focused on the non communicative individual who acted only on behalf of his own self-interest. These two different perspectives seem a reflex of the contrasting personalities of the two mathematicians. Von Neumann was the kind of academic who enjoyed having meetings and discussions of science with his colleagues.

He exchanged papers and ideas with other researchers, who he often helped solving specific problems they couldn't outrun. And his book on the theory of games was even written with the collaboration of a man of a different field, the economist Oskar Morgenstern. Nash, on the contrary, was the kind of man that liked to work alone, without joining any school of thought and without the orientation or collaboration of other researchers. Indeed he didn't even read or study other authors, preferring to construct his theories from ground and to discover the knowledge by him, without "trusting" in other people.

Curiously the concept of non-cooperative equilibrium turned out important to solve cooperative games as well. In his PhD dissertation, after presenting an application of the equilibrium to a three-man poker game, Nash claims that "a less obvious type of application is to the study of cooperative games (…) [where] players can and will collaborate as they do in the von Neumann and Morgenstern theory". Indeed, cooperative games can simply be solved by reducing the process of negotiation to a non-cooperative model, as Nash (1953) illustrates in *Two-person cooperative game*, an article published a few years later.

In this paper Nash presents a simple negotiation model between two distinct players. At the first stage of the model, each player establishes the strategy he will pursue if the two fail to reach an agreement (or, in other words, each player defines a threat that will be executed in case they fail to cooperate with each other). To guarantee the credibility of the threats players may sign a legal contract where they compromise to them. At the second stage of the game (the negotiation stage) each player demands a minimum level of payoff / utility for himself and, if the sum of their two "demands" can be attained within the rules of the game, they both get what they demanded. Otherwise they are forced to execute their threats.

Solving this problem by backward induction, Nash finds that there are plenty equilibria at the second stage of the game, but only one of them is stable and relevant. Because the procedures used in the paper are complex, extensive and require mathematical and topological tools, we cannot expose here his original proof. But we can at least explain the main economic reasoning behind the negotiation process. Each player, in order to maximize the expected utility he will get at the end of the game, chooses the amount of utility demanded taking into account that a greater demand increases the payoff obtained under cooperation, but it also decreases the probability of reaching an agreement. He faces, hence, a trade-off between the dimension and the consistency of his demand. At the solution, players choose the strategies that optimize the product of their net utility gain from cooperating. This idea was already present in *The Bargaining Problem* (Nash, 1950).

Obviously the “demands” or utility levels negotiated crucially depend on the threats that players define at the first stage of the game. But once we have determined the solution at the second stage, we can solve the first stage as any other non-cooperative game: at the equilibrium, any player simply chooses the threat that optimizes his expected utility holding fix the strategy of the other player.

Nash has thus provided us a strong model to investigate the distribution of the profits of a cartel. This distribution results from a bargaining process between firms, who demand the greatest possible share of total profits without jeopardizing their cooperative relation. If the firms are not able to legally compromise to a specific strategy, the threat executed when the agreement is not reached is simply the usual non-cooperative oligopoly solution. Therefore, a very important result is that the distribution of the profits of the cartel critically depends on the profits that firms would earn if they didn’t collude at all.

The ideas behind the *Two-Person Cooperative Game* have shown that some components of a collusion game can only be solved using a non-cooperative equilibrium concept, such as the negotiation of the distribution of total profits, where the gain of one firm is the loss of the other. As a result, this paper allowed us for the first time to approach collusion, at least in part, as a non-cooperative behavior. Although this idea may seem counter-intuitive, remark that the firms of a cartel do not cooperate and coordinate their strategies on behalf of their common good, but on behalf of their self-interest only. And thus there are inevitably some stages of a collusion game where firms must act uncooperatively.

Many other researchers have ever since applied the Nash equilibrium to solve cooperative games. Among them we are specifically interested in the work of Reinhard Selten, a German economist born on 1930 whose contributions to collusion theory remain essential for current research and, in particular, for the analysis of partial collusion (cartels where only some firms of the industry collude). In his classic paper *A Simple Model of Imperfect Competition where 4 are few and 6 are many*, Selten (1973) investigates the decision process through which companies decide whether or not to participate in a cartel. This decision is another move in a collusive game where firms act only on behalf of their self-interest, disregarding any impact of their decisions on other firms. For that reason, as Nash did it in the analysis of the bargaining process, Selten studies the decision of entering a cartel by reducing the collusive game to a non-cooperative form (where cooperative behaviors are modeled as moves in a non-cooperative game).

In his paper, Selten proposes a model composed by three different stages. Firstly, at the participation decision stage, firms simultaneously decide whether they want to enter the cartel bargaining. Then, the firms who decided to participate at the cartel bargaining stage communicate with each other and propose a quota system that defines the maximum levels of production. Only the firms who agree on an identical proposal are able to form a coalition and to sign an enforceable contract restraining their production levels. Finally, at the supply decision stage, firms produce and sell the output taking into account the maximum quotas they have previously defined. As always the model is solved by backward induction. However, because the third stage is a simple oligopoly game with production constrains and since the bargaining process has already been discussed, we will focus here on the solution of the first stage.

In order to understand the economic reasoning underlying the participation decision, consider an industry composed by n equal firms and suppose that a fraction p of those firms form a cartel. A first important proposition is that, everything else equal, the firms outside the cartel have greater profits than the firms inside, since the outsiders are able to deviate from the cooperative strategy in order to increase their own payoffs. A second proposition is that the greater is the fraction p, the greater are the profits of the firms of the cartel, since a larger number of players are cooperating for mutual gain. Given these two propositions, a firm entering a cartel faces two different effects: on the one hand the firm may decrease its payoff, because it is now an insider instead of an outsider; on the other hand, the firm may increase its payoff, once the total number of cooperating firms has risen.

With this type of reasoning Selten proves that, in industries with a small number of firms, the best strategy for each company is always to participate in the cartel, because the second effect is much stronger. In opposition, when the size of the industry is very large, all firms gain with the creation of a cartel but, if possible, they prefer to remain outside of it. As a result the optimal behavior consists in carrying a mixed strategy, according to which a firm decides to participate with a certain probability and not to participate with another probability.

Using linear demand and cost functions and a symmetry assumption, the author obtains an interesting numerical result: in industries with 4 or less firms, the whole industry always forms a cartel; in industries with 6 or more firms, it is very unlikely that any cartel will be created; in industries with 5 firms there is an intermediate situation where some firms join the cartel and others don't. This clarifies why 4 are few and 6 are many.

The results achieved by Nash and Selten through the application of the non-cooperative equilibrium to cooperative games were critical to the development of collusion theory. We are now not only able to determine how the total profits of a cartel are distributed between its members, but we can also establish in any industry which firms are going to collude and with what probability. Indeed, the many contributions of Nash and Selten to game theory earned them in 1994 the Nobel Memorial Prize in Economic Science, shared also with John Harsany, another game theorist.

Despite the use of the non-cooperative equilibrium to obtain new important results in collusion theory, it is important to remark that we are still modeling cartels as cooperative games in the sense that firms communicate and sign enforceable contracts. Therefore an important question arises: is it possible that firms collude without neither talking nor making any explicit agreements with each other?

1.5. The rise of non-cooperative collusion

Due to the severe effects of collusion on social welfare, most advanced market economies have created competition authorities and antitrust laws that forbid any sort of agreements or combinations between firms to conspire against free competition. In United States the first antitrust law was the Sherman Act published in 1890, followed by the Federal Trade Commission Act and the Clayton Act in 1914, as well as by many other state antitrust laws. In Europe the general rules on cartels are present in the Treaty of Rome signed in 1958 and in the Treaty on the functioning of the European Union in 2009, although each country has its own competition authority and specific directives to regulate collusion.

The fact that cartels became so tightly regulated with increasing sentences makes us wonder if firms are still able to successfully cooperate nowadays. Indeed, in the specific case of United States, collusion is a felony crime pursued by the Department of Justice and by the Competitive Bureau of Federal Trade Commission and it is punished not only with substantial fees, but also with prison sentences. Unfortunately, in the same way that not even life imprisonment or penalty death succeeds to prevent some hideous murders in several countries, there are always people willing to take chances and to attempt to make illegal cooperation agreements to increase their profits.

Nevertheless, there is no doubt that cooperative collusive activities are increasingly harder to sustain, largely due to the new methods and tools used by competition authorities to detect them. One of the most important tools of the American Antitrust Division is the Leniency Program, initially implemented in 1978 and revised later in 1993, according to which the first member confessing his participation in a cartel is given immunity from any criminal prosecutions, as long as he fully cooperates with the Division and fulfills other specified requirements. After the implementation of the Leniency Program the detection of cartelized industries became much more efficient, since firms were racing to be the firsts revealing the cartel, even before any investigation was started. According to Scott Hammond, the director of Criminal Enforcement of the Antitrust Division in 2000, “the United States' Corporate Leniency Program ("Amnesty Program") has been responsible for detecting and cracking more international cartels than all of our search warrants, secret audio or videotapes, and FBI interrogations combined." And its success was such that similar programs were applied in many other countries in the world.

Given the active and effective methods of the competition authorities to detect violations of the antitrust law, one would expect that few collusive practices could survive today, especially in countries like United States. However, empirical evidence suggests that prices and quantities transacted are still too far away from the non-cooperative levels in many oligopolistic industries. The only possible explanation is that firms found some way to, without communicating, make implicit or tacit agreements to keep prices high, restrain production levels and not to steal costumers from each other. This is the rise of non-cooperative collusion.

If the hypothesis of non-cooperative collusion is to be sustained, there must be some equilibrium strategies for which firms are able to collude without meeting, talking and without making any type of enforceable agreements or side payments. Such strategies cannot exist, of course, in a one-period static game. In this case, if the firms of the industry attempt to collude, the best strategy for every firm is to deviate from collusion and to collect a greater payoff. And because the game lasts only one period, there is no mechanism available to punish the firms who deviate. Hence the analysis of non-cooperative collusion only makes sense when the players interact repeatedly along time, so that any firm who deviates from the informal cartel can be punished by the others in the following periods.

The idea that a non-cooperative cartel may arise in dynamic environments was discussed for the first time by George Stigler (1964) on *A Theory of Oligopoly* and it was developed later by James W. Friedman (1971), who proved the existence of a non-cooperative "collusive" equilibrium in infinite supergames. By using the concept of a supergame (which consists in a sequence of ordinary static games played over time) Friedman introduced the role of dynamics in collusion theory and proved that when players are sufficiently patient regarding future payoffs (that is, when their discount factor is sufficiently large), a tacit cartel can be sustained in equilibrium.

To illustrate this point, consider an industry where each firm plays the collusive strategy in every period, as long as all the other firms did the same in the past periods; howsoever, if at any moment of time a firm deviates from the collusive path, the whole industry reverts to the competitive equilibrium forever. In this scenario any rational firm is willing to play the collusive strategy as long as the long run gains from colluding exceeds the "one shot" gain from deviating today. In other words, non-cooperative collusion can be played in equilibrium if the following incentive compatibility constrain (ICC) is verified:

$$\left(\pi_i^C - \pi_i^P\right)\frac{\delta}{1-\delta} \geq \pi_i^D - \pi_i^C, \tag{ICC}$$

where π_i^C is the profit received by firm i under collusion, π_i^P is the profit of the punishment phase (equal to the competitive profit), π_i^D is the profit earned by firm i when it is the only one deviating from the collusive path and δ is the discount factor. The left handed side of the ICC is the present value of all future gains of the cartel for firm i and the right handed side of the equation corresponds to the one shot gain from deviation. It can be shown that when the discount factor approaches the unit, the incentive compatibility constrain is always verified, proving that non-cooperative collusion is always possible if firms are patient enough about future payoffs.

This kind of supergame strategies proposed by Friedman, which would be later known as trigger strategies (since any deviation from the collusive path "triggers" the competitive equilibrium forever), revolutionized completely the economic literature on collusion. The remarkable work of the author led economists to start investigating collusion as a "pure" non-cooperative game and to analyze new important issues. For example, there are currently countless papers identifying the industries where informal collusion is more likely and determining how it can be avoided.

Some years later, Dilip Abreu (1984) proposed the substitution of trigger strategies by optimal penal codes in the analysis of infinitely repeated game with discounting. He argued that after a firm deviates from the collusive path, it would be much more efficient if all the other firms of the industry, instead of reverting to the competitive equilibrium, set their strategies to minimize the profits of the deviator. This way, by punishing in the hardest possible way any firm who deviates, the informal cartel becomes much easier to sustain because firms are afraid of being punished (this explains the term "optimal penal code"). In turn, if any of the firms deviates from the punishment phase, a new punishment would be restarted, targeting this time the new deviator.

Optimal penal codes are undoubtedly the most efficient strategy to sustain an informal cartel. And they turned out so important to the economic literature, that nowadays it is difficult to publish any article on informal collusion without at least mentioning them. Nevertheless, these strategies seem to move a little away from the notion of a non-cooperative game, as it is very unlikely that firms are able to coordinate so complex strategies without communicating with each other. In fact, it is even a little naive to suppose that every firm of the industry will minimize the payoffs not only of the firms who deviate from the collusive path, but also of the firms who didn't participate in the punishment phase, when they have never explicitly agreed to do so. For that reason we consider that trigger strategies are the apogee of the analyses of collusion as a non-cooperative game.

Finally we would like to make a last remark regarding informal cartels. We have shown that historical changes in the antitrust regulation considerably increased the importance of tacit collusion, which economists started modeling as a non-cooperative game. Yet informal cartels already existed even before collusive practices were forbidden by law. In a paper about non-cooperative collusion and imperfect information, Edward Green and Robert Porter give an example of the American rail freight industry as a cartel maintained in the 1880s (prior to the Sherman Act), whose firms played some kind of trigger strategy. However, because the demand of this industry was extremely volatile, the firms didn't know if a drop in market prices were the result of a fall in demand or of some firms deviating from the collusive path. Although we are not interested in the specifics of the paper, the uncertainty faced by the firms of the cartel clearly shows that they did not communicate with each other and hence this was, in fact, a non-cooperative cartel.

Informal cartels do not only exist for a long time, but they are also present in our reality more than we can imagine. Everywhere there are firms, sellers, employers, unions, countries and all the sort of economic agents making implicit agreements not to compete with each other, always at the cost of some third party left outside. And because these agreements are not written and not even discussed, most of the times it is almost impossible to detect them. Thus it is extremely important to use the appropriate game theory tools to investigate this evil and persistent phenomenon present in market economies.

1.6. Conclusions

Since the creation of a theory of games, new mathematical tools have been available to perform a more rigorous and formal analysis of collusion, contributing therefore to the increase of our scientific knowledge on the subject.

When the new paradigm of game theory was initially created by John von Neumann and Oskar Morgenstern, it focused mainly on cooperative games where players form coalitions and the solutions were obtained using the principle of domination. At this point, game theory could be applied to industrial organization to determine the different possible combinations of cartels that could arise in any industry, the strategies played by each coalition and the payoffs obtained. Unfortunately the concept of domination embraces too many solutions and is not able to explain other important issues in the analysis of cartels.

A few years later John Nash created a much more selective concept of solution for non-cooperative games, the non-cooperative equilibrium, which turned out important to solve cooperative games as well. In fact, Nash illustrated how the bargaining or the negotiation process between rational agents can be reduced to a non-cooperative game and solved afterwards. Similarly, his reasoning can be used to determine how the profits of a cartel are distributed among its members, once the repartition of profits crucially depends on a negotiation process between firms. Reinhard Selten also succeeded in applying the Nash equilibrium to cooperative games in the analysis of the participation decision in cartels. Due to his work we are now able to predict the likelihood of the formation of a cartel in any industry, as well as the probability with which a particular firm will participate in it.

The fact that the Nash equilibrium was so important to explain several aspects about collusion clearly shows that a cartel has many characteristics of a non-cooperative game, since its members always act only on behalf of their own self-interest.

Notwithstanding, collusion only started being investigated as a “pure” non-cooperative game when the increasing importance of tacit or informal collusion led economists as James Friedman and Dilip Abreu to explain how firms can sustain a cartel without meeting and communicating with each other. And the work they developed on trigger strategies and optimal penal codes was essential to solve new important problems.

We have thus shown that along the course of history the analysis of collusion has changed from the use of cooperative games to the use of non-cooperative games and such evolution had an enormous impact on our scientific knowledge about cartels.

Chapter 2

Theory of collusion in the labor market

2.1. Introduction

The modern tools of game theory and industrial organization have been widely applied to several research areas, as labor economics in more recent years, contributing to our growing understanding of the most complex economic phenomena. Among the many mysteries still unsolved involving the particular characteristics and frictions in the labor markets is the evidence that, in certain industries, employers have enough market power to sustain low levels of employment and to pay to workers considerably below their marginal productivity. In this respect, Bhaskar, Manning & To (2002) have shown that oligopsonistic and monopsonistically competitive structures in labor markets are, indeed, the only explanation for many empirical facts, as the dispersion of wages among workers with similar skills and the positive impact of minimum wages on employment. And according to Manning (2003), *"labour markets are 'thin' in the sense of there being few employment opportunities available at any moment in the immediate geographical locality of a worker"*, largely due to search frictions and heterogeneity of jobs. In this chapter we sustain the hypothesis that collusive activities in the labor markets may be an additional important source and explanation for the low degree of competition observed.

When we use the term collusion in the labor market we mean collusion in the demand side of the market, which may take the form of cooperative agreements between employers to reduce wages and employment levels, no-solicitation agreements and any other type of pacts to conspire against the worker and to distort free competition. In what follows, we briefly discuss three important empirical examples of those agreements.

In 1997, an employee of Exxon Mobil Corporation, Roberta Todd, initiated a lawsuit alleging that the company was able to save over 20 million dollars in annual wages paid to managerial, professional and technical employees, due to cooperative interactions between the firm and fourteen other oil companies, as BP, Shell and Chevron. The collusive practices denounced included the conduct of surveys about past and current salary information and future salary budgets, exchanges of large amounts of detailed information between the firms and frequent meetings between the human resource departments to discuss current and future wage budgets. And even though the lower court initially declined the claim, the court of appeals confirmed latter that this was, in fact, a violation of the antitrust laws.

In 2006 the two registered nurses Pat Cason-Merenda and Jeffrey A. Suhre brought a lawsuit on behalf of all registered nurses employed between 2002 and 2006 in several hospitals and medical centers in the Detroit Metropolitan Statistical Area. Those hospitals were accused of conspiring to depress the wage levels of registered nurses in the context of a national nurse shortage, by exchanging detailed and non-public information about remunerations through meetings, telephone conversations and written surveys. In 2009 a settlement agreement was reached with St. John's Health System, who has agreed to pay a compensation of over 13.5 million dollars.

In September 24, 2010, the Department of Justice of the United States enforceable the high technology companies Google, Apple, Adobe, Intel, Intuit and Pixar to stop entering into no-solicitation agreements, in which they compromised not to steal workers from each other. Such contracts were responsible for restraining the wages of high skilled workers and for reducing access to better job opportunities and they constituted thereby an anti-competitive conduct. Unfortunately, no-solicitation agreements appear to have been restarted in 2011, when some high skilled employees claimed that the "cold calls" offering better payments and working conditions have ceased. This led the software engineer Siddharth Hariharan to fill a lawsuit against the previously mentioned companies plus Lucasfilm, in which he accuses the companies for conspiring against free competition and demands a compensation exceeding 25 thousand dollars.

Although these three cases alone provide strong evidence of the temptation of firms to reduce competition in the labor market, some other examples could be given. For instance, in 2012 it was not for the first time that the union of American football players sued the National Football League for fixing a secret salary cap, claiming damages of about one billion dollars.

In fact there seems to be evidence of collusion in the labor market even in the middle age, as Peters (2010) discusses in a recent paper about the reactions of different labor markets to the Black Death in the fourteenth century. The author found that while in the Western Europe the fall in labor supply increased the wages, living conditions and political rights of the peasantry, the opposite occurred in the Eastern Europe and in the Middle East, where landlords succeeded to collude and to force peasants to supply more unpaid work. The paper discusses further the economic reasons that enabled the two last geographic areas to sustain collusion.

The damage of cartels on social welfare can be so severe that most advanced market economies have currently some form of legislation and authority to regulate collusive activities. In United States the first ever federal antitrust law, the Sherman Act, was created in 1890 and it stated in its first section that *"every contract, combination in the form of trust or otherwise, or conspiracy, in restraint of trade or commerce among the several States, or with foreign nations, is declared to be illegal"*. In the meantime other antitrust laws have been created, as the Federal Trade Commission Act and the Clayton Act, both published in 1914, besides several state antitrust laws. All these acts constitute criminal laws prosecuted by the Department of Justice and by the Bureau of Competition of the Federal Trade Commission and their violation have been punished with growing fines and prison sentences.

In Europe most countries have their own competition authorities, but general rules on competition have been provided by the Treaty of Rome since 1958 and by the Treaty on the functioning of the European Union after 2009, whose first paragraph of the article 101 states that *"The following shall be prohibited as incompatible with the internal market: all agreements between undertakings (...) which have as their object or effect the prevention, restriction or distortion of competition within the internal market"*.

Nevertheless, when we actually analyze the cartels under the investigation of antitrust authorities, we observe that almost all are involved in collusive agreements to fix prices, limit production or divide markets, while the investigation of collusion in the labor market is extremely rare, perhaps because competition policy is traditionally more concerned with the welfare of the consumer than that of the worker. Yet this questionable priority should not justify a poor supervision of the illegal contracts to fix wages or employment levels, especially because collusion in the labor market can damage the consumer in a very similar way to collusion in the final good market, as we will show later. Another reason that may partially explain why it is so hard to detect collusive activities in the labor market is the difference in the behavior of consumers and workers. While consumers who find themselves harmed by the cooperative actions of firms are always willing to report cartels to authorities, workers who find similar activities at their companies may prefer to remain silent in order not to risk losing their jobs or future career opportunities. But once again, the lack of investigation and the difficulty to detect collusion in the labor market should not mislead us to believe that those cooperative agreements seldom occur in reality. As Adam Smith said in 1776:

> *"We rarely hear, it has been said, of the combinations of masters, though frequently of those of workmen. But whoever imagines, upon this account, that masters rarely combine, is as ignorant of the world as of the subject. Masters are always and everywhere in a sort of tacit, but constant and uniform combination, not to raise the wages of labor above their actual price."*

Unfortunately, the analysis of collusion in the labor market has not been a primary concern of the economic literature to date. In the words of Manning (2010) *"we just don't know much about tacit collusion by employers because no-one has thought it worth-while to investigate in detail"*, although a few authors have addressed subjects somehow related. For example, Mukherjee, Selvaggi & Vasconcelos (2012) consider a principal-agent setting to study how exclusive employment contracts may increase the welfare of highly productive workers, by avoiding collusion between the principals; and Shelkova (2008) proposes that a non-binding minimum wage can be a focal point that coordinates tacit collusion between low-wages employers. These two articles not only fall on specific issues as exclusive employment contracts and non-binding minimum wages, but they also rely in particular scenarios as two-principals-two-agents contracts in the first case and a perfectly competitive product market in the second case. A more general model was presented by Bergès & Caprice (2008), who investigate how collusion in prices affects the wage and employment levels of qualified and unqualified workers.

Other relevant and recent papers can be found in a wider literature about collusion in the input market. González & Ayala (2012) propose a two-retailers-one-producer model to prove that collusion between the retailers to reduce the wholesale price of the input improves the profitability and stability of collusive strategies to raise the retail prices. In other words, they show that collusion in the input market promotes collusion in the downstream market, mainly because the two types of collusion make the punishment of deviating from the collusive path more severe. On the other hand, in a study by Christin (2009) it is presented another interesting result that the larger firms of an industry may sometimes collude to overbuy inputs, in order to exclude the smaller firms from the market. This strategy also reduces competition and harms consumers[2].

A major limitation of the extant economic literature is that most works address collusion in the final good market and collusion in the labor/input market separately, as if there was no connection between the two. In most economic models, firms are either assumed to cooperatively fix the price of the final good taking wages and other production costs as given, or to cooperatively determine the input price while the price of the final good is exogenously determined in a perfectly competitive market. Yet such assumptions may lead to spurious results. Indeed, when the American tobacco industry was convicted in 1946 for fixing the purchase prices of tobacco leafs (input) and the selling prices of cigarettes (final good), traditional oligopoly and oligopsony models were not able to explain how the companies of the cartel coordinated prices. On the contrary, Hamilton (1994) succeeded to fit the observed data using *"a model of joint oligopsony-oligopoly, which demonstrates that whatever market power the companies had in the cigarette and leaf market was unified."* It is hence the purpose of this chapter to study in detail the close relation between coordinating the price of the final good and the price of the input of production, which in our case is labor.

In this chapter we actually show that collusion in the labor market and collusion in the final good market are completely equivalent and have the same impacts on social welfare, causing prices to raise and output, employment and wages to fall. The conclusion that one type of collusion implies the other does not only improve our understanding of coordinated strategies in oligopolies and oligopsonies, but it also means that collusion in one of the markets can be used as a mechanism to conceal collusion in the other market. For instance, firms may choose to fix wages in order to avoid explicitly fix prices and to reduce the risk of getting caught by antitrust authorities. In addition, we show that the higher prices and lower wages in cartelized industries are the outcome of the elimination of the well-known *business stealing effect* and of the *labor force stealing effect*, which are presented and explained in detail. To

[2] Further topics about collusion in the input market are discussed in Dowd (1996).

ensure that the results obtained are robust we do not consider specific functional forms and main assumptions are kept as general as possible (with the exception of the one-input assumption, which is dropped in Chapter 3).

In Section 2.2 we present the firm maximization problem and the competitive equilibrium in a very simple model with competition in quantities, where the connection between the product market and the labor market can be easily illustrated. In Section 2.3 we present again the maximization problem and competitive equilibrium, but in a more robust model of price competition with differentiated products and differentiated job posts. In Section 2.4 we compute the collusive outcome, which is compared to the competitive equilibrium in order to derive the business stealing effect and the labor force effect. The impact of collusion on the main economic variables is then described in Section 2.5. In Section 2.6 we explain how collusion in the labor market may correspond to partial collusion in the product market and we analyze the effects of the collusive behavior on the firms outside the cartel, as well on other labor markets and industries. Section 2.7 discusses how the theory in this chapter can be applied to formal and informal cartels. Finally, Section 2.8 offers some concluding remarks.

2.2. A quantity competition model with homogenous products

Consider an industry where there is a finite number of n firms selling a final good in an oligopolistic market and hiring workers in an oligopsonistic labor market. The firms face a final good demand function $P = f(Q)$, which gives an inverse relation between the market price and the total quantity demanded ($f'(Q) < 0$), as well as a labor supply function[3] $W = g(L)$, according to which the wage increases with the total amount of labor offered ($g'(L) > 0$). The total quantity of good transacted and the total labor force employed are respectively equal to the sum of the quantity produced and labor used per firm ($Q = \sum Q_i$ and $L = \sum L_i$, $\mathrm{i} = \{1, \dots, \mathrm{N}\}$).

Consider further that each firm has a different production function that uses labor as the only input, $Q_i = h_i(L_i)$. While such assumption may seem somehow unrealistic and restrictive, once production technologies frequently employ different types of labor, capital and natural resources, one can think of labor as a composite input that gathers all the usual productive factors. The only implication of this is that when we study collusion in the labor market we implicitly assume that firms are able to cooperatively determine the price of all the productive factors included in the composite input. Another plausible explanation for the one input production function is to read our work as a short-run analysis where capital is fixed and, as result, it doesn't have any impact on optimal decisions about production and employment levels. Notwithstanding some may argue that the results could be significantly different when production functions have multiple inputs and firms can only successfully collude about the price paid to some (wages, for instance). We conduct such analysis in Chapter 3, where we show that the main qualitative results in this chapter remain valid, although the strategic interaction between firms is more complex and new effects and strategies occur. Therefore using production functions with one input allows us to prove some general results with simpler formal demonstrations, which may be particularly useful for those readers who do not wish to enter in more complex computations.

As a final assumption, in order to guarantee that our problem has a unique interior solution, we impose convexity to the cost function WL_i, which means that the labor

[3] In order to include unemployment in our model, one could consider instead a social labor supply function that accounts for the effects of labor unions and minimum wage policies.

supply function should not be excessively concave. The optimization problem of the firm is then to choose either the quantity produced (Cournot, 1838) or the amount of labor that maximizes the profit function $\pi_i = PQ_i - WL_i$. Due to the one input assumption, the production function establishes an exact relation between the quantity of good and the quantity of labor and so profits can be expressed in terms of one of the variables only. Consider profits as a function of the quantity produced:

$$\pi_i = f\left(\sum_{j=1}^{n} Q_j\right) Q_i - g\left(\sum_{j=1}^{n} h_j^{-1}(Q_j)\right) h_i^{-1}(Q_i). \qquad (2.2.1)$$

Given that the market price is a decreasing function of the quantity produced and that the cost function is convex, the profit function is concave and first order conditions can be used to obtain the maximum:

$$\frac{d\pi_i}{dQ_i} = 0 \quad \Leftrightarrow \quad f(.) + f'(.)Q_i = g(.)h_i^{-1\prime}(.) + g'(.)h_i^{-1\prime}(.)h_i^{-1}(.) \quad \Leftrightarrow$$

$$\Leftrightarrow \quad P + \frac{dP}{dQ}Q_i = W\frac{dL_i}{dQ_i} + \frac{dW}{dL}\frac{dL_i}{dQ_i}L_i. \qquad (2.2.2)$$

This optimal condition states that the firm produces up to the point where the marginal revenue is equal to the marginal cost. Because firms must reduce the price to sell more, the marginal gain from producing an extra unit of product is equal to the price earned with that unit minus the price reduction undertaken multiplied by the quantity that the firm was already able to sell at a greater price (the so called loss in the infra-marginal units). With regards to the marginal cost, producing an extra unit implies using a greater amount of workers that the firm must attract with a wage increase. Thus the cost of producing an extra unit of the final good is the wage paid to the new workers hired plus the wage variation times the number of workers that the firm already had before.

From equation (2.2.2) we obtain the best reply function $Q_i^{FOC}\left(\sum_{j\neq i}^{n-1} Q_j\right)$ and the profit is given by $max\{\pi_i{}^*, 0\}$, once the firm can always choose to leave the market to avoid any losses. Finally, the optimal quantity produced by each firm is:

$$Q_i^* = \max\{argmax\{\pi_i{}^*\}, 0\}.$$

Because the labor market has an oligopsony structure, every firm takes into account its power to partially control the wage level and so they choose to produce a lower quantity and to hire fewer workers in order to prevent the wage from growing too much, as we can see from the second term of the right-hand side of equation (2.2.2). In contrast, if the labor market was perfectly competitive, this term would be equal to zero and the optimal quantity produced would be larger.

The profit function could be rewritten as a function of labor only:

$$\pi_i = f\left(\sum_{j=1}^{n} h_j(L_j)\right) h_i(L_i) - g\left(\sum_{j=1}^{n} L_j\right) L_i. \tag{2.2.3}$$

This time the first order condition can be obtained by setting the derivative of the profit relative to the number of workers equal to zero. Note, however, that maximizing equation (2.2.1) with respect to quantities is equivalent to maximize equation (2.2.3) with respect to labor, since we only have expressed the profit function in terms of a different variable.

$$\frac{d\pi_i}{dL_i} = 0 \quad \Leftrightarrow \quad f(.)h_i{}'(.) + f'(.)h_i{}'(.)h_i(.) = g(.) + g'(.)L_i \quad \Leftrightarrow$$

$$\Leftrightarrow \quad P\frac{dQ_i}{dL_i} + \frac{dP}{dQ}\frac{dQ_i}{dL_i}Q_i = W + \frac{dW}{dL}L_i. \tag{2.2.4}$$

We conclude that each firm is optimizing profits when the gain of hiring an extra worker equals its additional cost. Because the extra worker raises the production capacity of the firm, its marginal gain corresponds to the market value of the new units produced minus the necessary price reduction multiplied by the quantity that the firm was already able to sell with a smaller labor force. The marginal cost of labor is the wage that must be paid to the additional worker plus the product of the number of workers already hired times the necessary wage variation to attract an extra worker.

From the first order condition in equation (2.2.4) we obtain the best reply function $L_i^{FOC}(\sum_{j\neq i}^{n-1} L_j)$. The profit is given by $\max\{\pi_i{}^*, 0\}$ and the optimal amount of labor is $L_i^* = \max\{argmax\{\pi_i{}^*\}, 0\}$. Once again, when deciding how many workers to hire in the labor market, the firm takes into account its power to influence the price of the final good. In fact, as long as the product market is not perfectly competitive, the second term of the left-hand side of equation (2.2.4) is not null and the firm measures the impact of hiring more workers in the production level and, thereby, in the final price.

The optimal conditions previously obtained can be used to get the final good supply and the labor demand functions. Solving equation (2.2.2) in order to the price gives the inverse supply function of the firm, $P = \phi_i(Q_i)$. The market final good supply is equal to the sum of the quantities produced by all firms together: $Q = \sum \phi_i^{-1}(P)$. In turn, solving equation (2.2.4) in order to the wage gives the inverse labor demand of the firm, $W = \theta_i(L_i)$, and the market labor demand is equal to the sum of the labor demanded by all firms: $L = \sum \theta_i^{-1}(W)$.

The non-cooperative equilibrium of the model corresponds to the market price, wage, individual quantities produced and number of workers hired by firm for which the product market and the labor market are in equilibrium. Solving the system of n best reply functions expressed in quantities gives the quantity produced by firm. Replacing these quantities in the production functions or solving the system of n best reply functions in terms of labor gives the number of workers hired by firm. Finally the market price and wage can be directly obtained from the final good demand and labor supply functions.

2.3. A price competition model with product differentiation

Although the previous quantity competition model was useful to provide some initial intuition about the relation between the labor market and the final good market, we believe that price competition usually represents a better approximation from reality, as in most industries firms are free to set the optimal prices to compete with their rivals. If the products sold and the jobs offered were homogeneous, we could use a Bertrand (1883) model to describe the interactions in the product and labor markets. But again, empiric evidence strongly suggests that most products are not completely identical and consumers are willing to pay higher prices for the goods that have the characteristics they most value. Similarly, jobs in the industry are usually seen by workers as heterogeneous, due to different working conditions, benefits, company values or even distance from home.

For these two reasons consider now price competition in the final good market with product differentiation and wage competition in the labor market with job differentiation. It is not important whether the product or job differentiation arises from an endogenous mechanism as it is described in the Hotelling model (1929) or in the circle model by Salop (1979). All it matters is that each firm faces an individual demand function that is decreasing with respect to its own price and increasing with respect to other firms' prices, $Q_i = f_i(P_i, P_{-i})$, and an individual labor supply that increases with the own wage and decreases with other firms' wages, $L_i = g_i(W_i, W_{-i})$. Naturally, market demand $Q = \sum f_i(P_i, P_{-i})$ is decreasing with respect to any price and total labor supply $L = \sum g_i(W_i, W_{-i})$ is increasing with respect to any wage. Once again, each firm faces a production function that uses labor as the only input, $Q_i = h_i(L_i)$, and the cost function given by $W_i g_i(W_i, W_{-i})$ is convex (that is, the labor supply function cannot be too concave).

At the non-cooperative equilibrium each firm chooses the price and the wage that maximize profits, taking the decisions of the other firms as given. Despite the absence of a direct relation between the price and the wage, the profit function $\pi_i = P_i Q_i - L_i W_i$ can still be written as a function of only one of these two variables. Consider first the profit function in terms of prices:

$$\pi_i = P_i\, f_i(P_i, P_{-i}) - h_i^{-1}\big(f_i(P_i, P_{-i})\big)\; g_i^{-1}\big(h_i^{-1}\big(f_i(P_i, P_{-i})\big), W_{-i}\big).$$

Since the profit function is concave, its maximum is attained at the point where the derivative of profits in order to the price is equal to zero:

$$\frac{d\pi_i}{dP_i} = 0 \quad \Leftrightarrow$$

$$\Leftrightarrow f_i(.) + P_i \frac{\partial f_i(.)}{\partial P_i} - h_i^{-1\prime}(.) \frac{\partial f_i(.)}{\partial P_i} g_i^{-1}(.) - h_i^{-1}(.) g_i^{-1\prime}(.) \frac{\partial h_i^{-1}(.)}{\partial f_i(.)} \frac{\partial f_i(.)}{\partial P_i} = 0 \ \Leftrightarrow$$

$$\Leftrightarrow \quad Q_i + P_i \frac{\partial Q_i}{\partial P_i} - \frac{\partial L_i}{\partial Q_i} \frac{\partial Q_i}{\partial P_i} W_i - L_i \frac{\partial W_i}{\partial L_i} \frac{\partial L_i}{\partial Q_i} \frac{\partial Q_i}{\partial P_i} = 0. \qquad (2.3.1)$$

According to equation (2.3.1), when the firm is maximizing its profit a small change in the price does not have any effect on profits. Indeed, if the derivative was positive it would be profitable to slightly increase the price, while if it was negative the opposite would be true. The first two terms of the left-hand side of equation (2.3.1) are the impact of a price variation in the revenue obtained in the goods market: when the price increases, the firm gains one more unit of money for each unit of product transacted and loses the market value of all the products that are no longer sold due to the price rise. The two last terms of the left-hand side of equation (2.3.1) are the effect of a price variation in the total costs incurred from hiring workers in the labor market. On the one hand, when the price is increased, less workers are required to satisfy the falling demand, allowing the firm to save on the wages paid to those workers. On the other hand, once the firm hires fewer workers, it is now able to hold all the necessary labor force at a lower wage level.

The best response function that comes from equation (2.3.1), $P_i^{FOC}(P_{-i}, W_{-i})$, defines the optimal price for firm i as a function of a vector of prices and wages from the other firms and the profit obtained is equal to $\max\{\pi_i^{FOC}, 0\}$. Analogously to the last section, when the firm sets the optimal price in the product market it takes into account its power to constraint the wage paid to the workers. Therefore, the firm sets a higher price than it would if the labor market was perfectly competitive, case in which the fourth term in the left-hand side of equation (2.3.1) would be null.

Now consider the profit function expressed in terms of wages:

$$\pi_i = f_i^{-1}\big(h_i\big(g_i(W_i, W_{-i})\big), P_{-i}\big)\, h_i\big(g_i(W_i, W_{-i})\big) - \ g_i(W_i, W_{-i}) W_i.$$

The corresponding first order condition is the following:

$$\frac{d\pi_i}{dW_i} = 0 \;\Leftrightarrow\; f_i^{-1}(.)h_i'(.)\frac{\partial g_i(.)}{\partial W_i} + \frac{\partial f_i^{-1}(.)}{\partial h_i}h_i'(.)\,\frac{\partial g_i(.)}{\partial W_i}h_i(.) - g_i(.) - \frac{\partial g_i(.)}{\partial W_i}W_i \;\Leftrightarrow$$

$$\Leftrightarrow\; P_i\frac{dQ_i}{dL_i}\frac{\partial L_i}{\partial W_i} + \frac{\partial P_i}{\partial Q_i}\frac{dQ_i}{dL_i}\frac{\partial L_i}{\partial W_i}Q_i - L_i - \frac{\partial L_i}{\partial W_i}W_i = 0. \qquad (2.3.2)$$

The two first terms of equation (2.3.2) describe the revenue gain obtained in the final good market when the firm raises the wage level in one unit: on the one hand, a greater wage attracts more workers that are able to produce more and the firm earns the market value of the extra production; on the other hand, to sell the additional production the firm must reduce the price, which decreases the revenue of all the units of the good that were already sold before. The two last terms of equation (2.3.2) correspond to the additional cost incurred in the labor market: when the wage increases in one unit, not only the firm has to pay an extra unit of money for every worker in the company, but it also attracts more employees to whom must be paid the whole wage.

From equation (2.3.2) we get the best response function that expresses firm i's optimal wage as a function of the wages and prices set by its rivals, $W_i^{FOC}(W_{-i}, P_{-i})$. As usual, every firm is free to leave the market to avoid losses and profits are thus given by $\max\{\pi_i^{FOC}, 0\}$. Finally note that a firm setting the wage in the labor market is aware of its ability to influence the price in the product market, as we can see from the second term of the left-hand side of equation (2.3.2) (this term is different than zero as long as the product market is not perfectly competitive).

The competitive equilibrium of this model consists in four vectors of n variables, the prices, quantities, number of workers and wages, for which every firm in the market optimizes profits given the decisions undertaken by its rivals.

The $4n$ equilibrium variables of the model can be obtained with $4n$ equations: n goods demand functions, n labor supply functions and n production functions that are exogenous to the model, as well as n best reply functions obtained from profit maximization. Although we derived n best reply functions in terms of prices and n best reply functions in terms of wages, the firsts are equivalent to the seconds, as they are closely related through the exogenous functions of the model.

2.4. Collusive outcome

So far we have studied how firms that produce a final good using labor as their only input make optimal decisions in a competitive scenario. In this section it is shown how the strategic interaction and main results change when firms build a cartel that optimizes joint profits. The results obtained here are crucial to understand the role of competition authority in the regulation of the labor market.

When firms compete in quantities and products are homogeneous, the collusive outcome corresponds to all firms producing together the monopoly quantity and setting the market price and wage that allows them to produce such quantity. If firms have equal and constant marginal costs, it doesn't matter how the cartel's total output is distributed between them. If instead firms have different linear cost technologies, only the most efficient ones should produce (unless the presence of capacity constrains requires less efficient firms to produce a part of the monopoly quantity). And if firms have different non linear cost technologies, the total production should be distributed between them in such a way that the marginal cost of producing an extra unit is equal for every firm. Despite the greater simplicity of working with competition in quantities, from now on we will only consider price and wage competition with differentiated products and job posts, which certainly characterizes better the majority of the industries.

For that reason suppose now that each firm produces a specific differentiated good that can only be achieved using the firm's production technology. Consider also that the degree of efficiency in the use of labor and the attractiveness of working conditions vary from firm to firm. In these circumstances it is no longer possible to define a global market price and wage that maximize aggregate profits. Instead, the cartel must define a specific price and wage per firm that takes into account how the product is valued by consumers, the efficiency of the cost technology and the attractiveness of the job. Therefore the vector of prices and wages that maximize the profits of the cartel are the solution to the following maximization problem:

$$Max_{P_i, W_i,\ i=1,\dots,n} \quad \pi = \sum_{j=1}^{n} P_j Q_j - L_j W_j$$

$$s.t. \quad \begin{cases} Q_1 = f_1(P_i, P_{-i}) \\ \dots \\ Q_n = f_n(P_i, P_{-i}) \end{cases}, \quad \begin{cases} L_1 = g_1(W_i, W_{-i}) \\ \dots \\ L_n = g_n(W_i, W_{-i}) \end{cases}, \quad \begin{cases} Q_1 = h_1(L_1) \\ \dots \\ Q_n = h_n(L_n) \end{cases}.$$

Although it is not possible to eliminate all the restrictions and to express profits as a function of either wages or prices, the optimization problem faced by the cartel can still be simplified to have one set of restrictions only:

$$Max_{P_i, W_i\ i=1,\dots,n} \quad \pi = \sum_{j=1}^{n} P_j\, f_j(P_i, P_{-i}) - g_j(W_i, W_{-i})\, W_j$$

$$s.t. \quad \begin{cases} h_1^{-1}\big(f_1(P_i, P_{-i})\big) = g_1(W_i, W_{-i}) \\ \dots \\ h_n^{-1}\big(f_n(P_i, P_{-i})\big) = g_n(W_i, W_{-i}) \end{cases}.$$

This particular specification of the problem allows us to withdraw some previous conclusions. Despite having $2n$ instruments available to optimize profits (n prices plus n wages), the cartel is constrained by n different restrictions. This means that actually the cartel has only n instruments to optimize profits, while the other n instruments are directly obtained from the firsts. Because the profit function is concave and the solution is unique, the cartel may choose prices and wages directly result from labor supplies, production functions and final good demands; or the cartel may set the optimal wages and prices are obtained from the demand functions, given the production levels of all the workers of the industry.

In other words, collusion in the goods market and collusion in labor market are completely equivalent when the firms that compete in the two markets are the same.

2.4.1. Business stealing effect

The optimization problem previously described can be solved using the Lagrange multipliers method, which consists in maximizing the following objective function:

$$L = \sum_{j=1}^{n} P_j\, f_j(P_i, P_{-i}) - g_j(W_i, W_{-i})\, W_j + \lambda_j \left[g_j(W_i, W_{-i}) - h_j^{-1}\left(f_j(P_i, P_{-i}) \right) \right].$$

The Lagrangian multipliers λ_j can be interpreted as the shadow price of labor, in other words, as the value that one worker operating in firm j has to the total profits of the cartel.

At the optimal interior solution the derivatives of the Lagrangian function with respect to the decision variables $(P_1, \ldots, P_n, W_1, \ldots, W_n, \lambda_1, \ldots, \lambda_n)$ are equal to zero. Thus, the first order conditions with respect to prices are:

$$\frac{\partial L}{\partial P_i} = 0 \;\; \Leftrightarrow \;\; Q_i + \sum_{j=1}^{n} \left(P_j - \lambda_j \frac{\partial L_j}{\partial Q_j} \right) \frac{\partial Q_j}{\partial P_i} = 0. \qquad (2.4.1)$$

From equation (2.4.1) it follows that when a firm increases the price in one unit, it not only earns one extra unit of money for each product sold and loses the mark-up (price minus marginal cost of production) of the final products that are not sold anymore $\left(\partial Q_i / \partial P_i\right)$, but it also accounts for the positive impact of fixing a higher price on the demand of the remaining firms $\left(\partial Q_j / \partial P_i\right)$, who get an additional revenue. Howsoever this last impact of increasing the price was not considered in the non-cooperative setting, as one can easily observe in the optimal condition in equation (2.3.1) rewritten in terms of the mark-up:

$$Q_i + \left[P_i - \left(W_i + L_i \frac{\partial W_i}{\partial L_i} \right) \frac{\partial L_i}{\partial Q_i} \right] \frac{\partial Q_i}{\partial P_i} = 0. \qquad (2.4.2)$$

This means that relatively to the non-cooperative scenario each firm has an extra incentive to raise the price, as it is aware of the positive effect it has on the production levels of the other companies. Actually, one of the reasons why prices tend to be lower in competitive markets is the existence of a business stealing effect. When firms are free to compete with each other they tend to reduce prices as an attempt to steal valuable market shares from their rivals. Because the cartel eliminates this business stealing effect, companies refrain from lowering prices, as that would injure the total profitability of the cartel.

2.4.2. Labor force stealing effect

A deeper analysis of equation (2.4.1) allows us to observe another effect that contributes to the rising prices of the cartel. As we can see in that equation, the net gain of selling an extra unit of good is the price minus the marginal cost of production, which corresponds to the additional amount of labor used multiplied by the value that each worker has to the cartel given its best alternative use (λ). To comprehend what is indeed the value or shadow price of labor, we must take a look to optimal conditions with respect to wages:

$$\frac{\partial \pi}{\partial W_i} = 0 \;\; \Leftrightarrow \;\; \sum_{j=1}^{n} \lambda_j \frac{\partial L_j}{\partial W_i} = L_i + \sum_{j=1}^{n} \frac{\partial L_j}{\partial W_i} W_j . \tag{2.4.3}$$

According to equation (2.4.3) the wage of any firm must be set at the level where the marginal gain – the increase in the entire labor force times the value that each worker has to the cartel – is equal to the marginal cost – the extra unit of money that must be paid per worker plus the wages that must be paid to the new workers hired. It is important to remark that, once again, the wage paid by firm i is determined taking into account not only the effect it has on the workers hired by that firm, but also on the cartel's entire labor force. To find an expression for the shadow price of the workers operating in firm i we solve equation (2.4.3) for λ_i:

$$\lambda_i \frac{\partial L_i}{\partial W_i} + \sum_{j \neq i}^{n-1} \lambda_j \frac{\partial L_j}{\partial W_i} = L_i + \frac{\partial L_i}{\partial W_i} W_i + \sum_{j \neq i}^{n-1} \frac{\partial L_j}{\partial W_i} W_j \quad \Leftrightarrow$$

$$\Leftrightarrow \quad \lambda_i = W_i + L_i \frac{\partial W_i}{\partial L_i} - \sum_{j \neq i}^{n-1} (\lambda_j - W_j) \frac{\partial L_j}{\partial W_i} \frac{\partial W_i}{\partial L_i} . \tag{2.4.4}$$

From equation (2.4.4) we conclude that if the cartel is maximizing joint profits, the value of the last worker employed by firm i (λ_i) must be equal to the cost of hiring him. But while in the competitive setting the cost of hiring an extra worker is, as we can see in equation (2.4.2), the wage paid to him plus the necessary wage variation to increase the labor force of the firm times the labor force employed ($W_i + L_i \times \partial W_i / \partial L_i$), in the collusive setting the cost of an additional worker corresponds to that same expression plus the cost of stealing labor from the remaining firms, who lose the difference between their shadow price and wages (third term in the right-hand side of equation (2.4.4)).

To understand this crucial difference note that when firms are competing they only care about their own labor force and so hiring an additional worker may imply stealing workers from other firms of the industry. Indeed, in the non-cooperative equilibrium there is a "labor force stealing effect" that motivates all firms to increase wages in order to attract workers employed in rival companies, causing employment and production levels to rise and prices to fall. Because the cartel eliminates the labor force stealing effect, companies become more reluctant in declining prices as they know that their marginal cost of production is now greater, once it includes the cost of stealing workers from other firms:

$$MgC_{Collusion} = \left[W_i + L_i \frac{\partial W_i}{\partial L_i} - \sum_{j \neq i}^{n-1} (\lambda_j - W_j) \frac{\partial L_j}{\partial W_i} \frac{\partial W_i}{\partial L_i} \right] \frac{\partial L_i}{\partial Q_i} >$$

$$> \left[W_i + L_i \frac{\partial W_i}{\partial L_i} \right] \frac{\partial L_i}{\partial Q_i} = MgC_{Free\,Market}.$$

Although we have proved that prices are higher under collusion due to the elimination of the labor force stealing effect, our definition of the shadow price of labor in equation (2.4.4) is still difficult to understand, as it is expressed in terms of the shadow price of the workers of the other firms. In order to find a more intuitive economic interpretation for the shadow price, we will assume for now that the cartel is symmetric, that is, all the firms have the same cost technology and face symmetric goods demand and labor supply functions. In this case, the price, wage and shadow price is equal for every firm and equation (2.4.3) can be simplified as follows:

$$\lambda_i \sum_{j=1}^{n} \frac{\partial L_j}{\partial W_i} = L_i + W_i \sum_{j=1}^{n} \frac{\partial L_j}{\partial W_i} \iff \lambda_i = W_i + \frac{L_i}{\sum_{j=1}^{n} \frac{\partial L_j}{\partial W_i}} \iff \lambda_i = W_i + \frac{L_i}{\frac{\partial L}{\partial W_i}} \iff$$

$$\iff \lambda_i = W_i + L_i \frac{\partial W_i}{\partial L}. \qquad (2.4.5)$$

We conclude that in a cartel composed by symmetric firms, the cost of hiring an extra worker is the wage paid to him (W_i) plus the necessary wage variation to increase the labor force of the cartel in one unit $(\partial W_i / \partial L)$ (instead of the wage variation required to increase the labor force of the firm in one unit, $\partial W_i/\partial L_i$) times the number of workers employed at firm i (L_i). Indeed, when firms are engaged in a symmetric cartel a worker stolen from other firms has absolutely no additional value, since the total labor force of the cartel remains constant and the value of the worker is the same regardless of the company he is employed in (due to the symmetry assumption). Hence the shadow price λ_i refers to the value of hiring an additional worker that was not operating in the industry yet. And the cost of such worker includes the necessary wage variation to increase the total labor force in one unit $(\partial W_i/\partial L)$.

Now that we are aware of what the shadow price of labor is when the cartel is optimizing profits, we can apply the symmetry assumption to equation (2.4.1) and replace λ_i by equation (2.4.5):

$$Q_i + \sum_{j=1}^{n}\left(P_j - \lambda_j \frac{\partial L_j}{\partial Q_j}\right)\frac{\partial Q_j}{\partial P_i} = 0 \;\;\Leftrightarrow\;\; Q_i + \left(P_i - \lambda_i \frac{\partial L_i}{\partial Q_i}\right)\sum_{j=1}^{n}\frac{\partial Q_j}{\partial P_i} = 0 \;\;\Leftrightarrow$$

$$\Leftrightarrow \;\; Q_i + \left[P_i - \left(W_i + L_i\frac{\partial W_i}{\partial L}\right)\frac{\partial L_i}{\partial Q_i}\right]\frac{\partial Q}{\partial P_i} = 0. \qquad (2.4.6)$$

Given that $\partial W_i/\partial L$ is greater than $\partial W_i/\partial L_i$, we once again conclude that the marginal cost of production is larger in the collusive scenario, which encourages the firms of the cartel to raise prices. It is important to remark at this point, however, that such conclusion does not depend on the assumption of symmetry, which was only imposed for exposition purposes. Finally, to prove that it takes a greater wage variation to raise the entire labor force of the cartel in one unit $(\partial W_i/\partial L)$ than to raise the labor force of the firm in one unit $(\partial W_i/\partial L_i)$ is straightforward, given that the labor supplied to any firm decreases with the wages set by its rivals and given that the total labor supply is equal to the sum of the labor supplies of all firms:

$$\frac{\partial L}{\partial W_i} = \frac{\partial L_i}{\partial W_i} + \sum_{k\neq i}^{n-1}\frac{\partial L_k}{\partial W_i}.$$

$$\sum_{k\neq i}^{n-1}\frac{\partial L_k}{\partial W_i} < 0 \;\;\Rightarrow\;\; \frac{\partial L}{\partial W_i} < \frac{\partial L_i}{\partial W_i} \;\;\Leftrightarrow\;\; \frac{\partial W_i}{\partial L_i} < \frac{\partial W_i}{\partial L}.$$

2.5. Impact of collusion on the main economic variables

In the previous sections we compared the optimal condition for the price of a particular firm i under free competition with the optimal condition under collusion. We asserted that, holding fix the prices of the rest of the industry, the price charged by firm i is set at a higher level in the collusive setting in order to eliminate the business and labor force stealing effects. However, we must now take into account that, under collusion, the prices of the remaining firms of the industry are increased as well and, as a result, the optimal price charged by firm i is even higher than it would be if the prices of the other firms remained constant[4]. This clearly shows that the formation of a cartel raises the prices of every single firm in the industry.

Similarly, if we hold fix the wages of the other firms, the higher price set by firm i under collusion directly determines that the wage paid must be lower, due to the strong connection between the labor market and the final good market. But, once again, because the other firms in the cartel are also pressured to decrease the wage, the optimal wage paid by firm i is fixed at an even lower value in the collusive scenario. An alternative way to prove that collusion reduces the wage levels of the whole industry is to repeat the procedures in the previous sections for the optimal conditions with respect to wages.

Once it was determined that the formation of a cartel augments all prices and reduces all wages of the industry, it follows directly from the market demand function and total labor supply function that the total quantity transacted and the total employment level must fall. In turn, this sustains that collusion in the labor market and collusion in the final good market have always a negative impact on the consumer and worker's welfare.

It should be highlighted, though, that when a cartel is composed of firms which are sufficiently asymmetric, optimal behavior may require some of them to augment the production level and to hire more workers. In fact, suppose that in a cartelized industry there is one firm much more efficient than the others and which produces a good preferred by most consumers. In such case the cartel will artificially increase

[4] It follows from equations 7 and 8 that the optimal price of one firm is a positive function of the other prices.

considerably less the price of that firm than the price of the others and, consequently, it may be the case that the efficient firm is required to produce more and to hire more workers. Nevertheless, this would be an exceptional case and, on average, firms would still decrease the production level and employ fewer workers.

2.6. Partial collusion

The previous analysis of collusion focused on industries where the firms in the supply side of the product market coincide with the firms in the demand side of the labor market. But in reality this perfect match is not common, either because different firms of the same industry have access to different labor markets (case in which collusion in the labor market causes partial collusion in the product market) or because the same labor market supplies workers to different industries (case in which collusion in the product market leads to partial collusion in the labor market). This section covers the former case.

Consider an industry where $n+m$ firms produce and sell a differentiated product in the final product market, competing with each other in prices. As usual, each firm faces an individual good demand function that decreases with its price and increases with the price of the rivals, $Q_i = f_i(P_i, P_{-i})$, as well as a production function that uses one type of labor as the only input, $Q_i = h_i(L_i)$. Suppose further that the industry can be decomposed in two sets of firms that set wages and hire workers at two distinct labor markets, n firms at labor market A and the remaining m firms at labor market B. The reason why different firms have access to different labor markets may either be their concentrated location at different industrial zones or because their production functions require different types of specialized workers. The labor supplied to each firm is increasing with respect to the own wage and decreasing with respect to the wages of the rivals located at the same labor market, $L_i^A = g_i(W_i^A, W_{-i}^A)$ and $L_i^B = g_i(W_i^B, W_{-i}^B)$.

It is now our concern to study the economic effects of collusion in one of the labor markets. Suppose that the n firms located at labor market A decide to set wages cooperatively, which as we already know directly determines their price levels. When colluding, the optimizing behavior of the n firms is characterized by equation (2.6.1):

$$Q_i^A + \left[P_i^A - \left(W_i^A + L_i^A \frac{\partial W_i^A}{\partial L^A}\right)\frac{\partial L_i^A}{\partial Q_i^A}\right]\frac{\partial Q^A}{\partial P_i^A} = 0. \qquad (2.6.1)$$

As always the n firms of the cartel refrain from stealing business and labor force from each other and the result is that they all fix greater prices and smaller wages. Notwithstanding the industry is not composed only by the n firms involved in the cartel, but also by other m firms competing in the final product market who are going to react to the rising prices. Because the later firms are not colluding, their non-cooperative optimizing behavior is characterized by equation (2.3.1):

$$Q_i^B + \left[P_i^B - \left(W_i^B + L_i^B \frac{\partial W_i^B}{\partial L_i^B}\right) \frac{\partial L_i^B}{\partial Q_i^B}\right] \frac{\partial Q_i^B}{\partial P_i^B} = 0 \ \Leftrightarrow$$

$$\Leftrightarrow \ P_i^B = \left(W_i^B + L_i^B \frac{\partial W_i^B}{\partial L_i^B}\right) \frac{\partial L_i^B}{\partial Q_i^B} - \frac{Q_i^B}{\partial Q_i^B / \partial P_i^B} \ \Leftrightarrow \ P_i^B = MgC_i^B - \frac{Q_i^B}{\partial Q_i^B / \partial P_i^B} \ \Leftrightarrow$$

$$\Leftrightarrow \ P_i^B = \frac{\partial Total\ Cost_i^B}{\partial Q_i^B} - \frac{Q_i^B}{\partial Q_i^B / \partial P_i^B}. \tag{2.6.2}$$

Equation (2.6.2) is the best reply function for any of the m firms at labor market B, which depends on the prices of the firms of the cartel through the quantity produced Q_i^B. Using equation (2.6.2) it is possible to prove that under convex cost functions and concave and additive demand functions, the best reply to the non-cooperative firms is to produce more at a higher price (see proof in Appendix A).

Intuitively, when the prices of the firms belonging to the cartel go up, the other firms in the industry face a greater demand and sell more, which encourages them to increase prices for two reasons: in the first place, producing more implies a greater marginal cost that must be offset by a greater price; secondly, the marginal gain of increasing the price is now multiplied by a larger number of units of the final good.

In addition, since all the m non-cooperative firms face now a greater demand, they have to increase the wages in order to attract more workers at the labor market B, in order to meet the new production levels. Note that the wage defined by each firm is indeed an increasing function of the quantity produced, $W_i^B = g_i^{-1}\left(h_i^{-1}\left(Q_i^B\right), W_{-i}^B\right)$.

In conclusion, the effects of the creation of a cartel in labor market A go far beyond the fall in wages and employment at that market. The most important effect is probably the increase in prices not only of the cooperative firms, but also of the whole industry. Now, once total demand falls with prices, the total quantity transacted is reduced to the detriment of consumer welfare. In addition, and perhaps more interestingly, collusion in labor market A has a positive externality effect on wages and employment in labor market B, once these two markets are connected by the same industry. Indeed when the firms at labor market A collude, some of their production level is transferred to the other firms of the industry, who must hire more workers at greater wages.

The analysis in this section could be prosecuted by considering that labor market B also supplies workers to other industries, who would be able to attract fewer workers and who would be forced to produce less at higher prices. Thus the general conclusion is that collusion in a specific labor market may have negative effects on prices and quantities on several industries but, simultaneously, positive effects on wages and employment on other labor markets.

2.7. Some notes on the cooperative nature of collusion

In the previous sections we have studied the effects of the creation of a cartel that maximizes joint profits by fixing the optimal wages in the labor market or, equivalently, by setting the optimal prices in the final product market. In this section we wonder if the assumption of joint profit maximization is reasonable, which may depend on whether the game played is cooperative or non-cooperative.

Suppose firstly that firms play a cooperative game, by which we mean that they are able to communicate with each other, negotiate enforceable agreements and make side payments. In this case, if joint profits are not being optimized, firms can always negotiate a better deal and use side payments to guarantee that everyone gains. And because the new contract is enforceable, they are willing to do such deal until joint profits are at its maximum. Although cooperative behavior is harder when formal cartels are illegal, it is always possible to communicate and to make side payments "off the record", while credible threats can be used to enforce the negotiated agreements. Therefore, it is extremely important to regulate and to investigate whether there is collusion in the labor market, in order to avoid cooperative behaviors that lead to the maximization of joint profits along with the negative consequences on social welfare.

Unfortunately, even if the competition authority is able to compel firms not to cooperate explicitly, an informal cartel can still be sustained using, for example, the trigger strategies in Friedman (1971) or the optimal penal codes in Abreu (1984). In the particular case of trigger strategies it can be shown that, as long as every firm is strictly better off when colluding, there are discount rates lower than one for which collusion can be sustained in equilibrium at every stage of an infinite super game (see Appendix B). However, because the game is now non-cooperative and side payments are not possible anymore, at least some symmetry is required to guarantee that joint profit maximization improves the profits of all firms. Hence the theory presented in this chapter can also be applied to non-cooperative games when the firms of the informal cartel are not too heterogeneous. Finally it is important to recall that joint profit maximization is not the only non-cooperative equilibrium, as it is proved by the *Folk* theorem.

2.8. Conclusions

We have formulated a general theoretical framework to investigate collusion in the labor market and we believe that the results obtained are not only relevant for economic policy purposes, but they are also surprising and unexpected in some degree, contributing to a better understanding of some interactions observed in the labor markets.

The first temptation for anyone who addresses this subject for the first time is probably to imagine that any agreement between employers to decrease wages reduces the marginal cost of production and, as a result, encourages firms to sell more at lower prices, making consumers better off. Yet this reasoning would only make sense if the fall in wages was caused, not by the cooperative action of firms, but by an exogenous shock in the external environment of the industry. For instance, if a mass immigration of highly-skilled workers reduced the wage level, firms would indeed be able to hire more workers at a lower cost and so they could produce more and charge a lower price for the final good. This is not the case, however, when the fall in wages is caused by an endogenous mechanism as collusive agreements. As we have established in this chapter, in the absence of external exogenous shocks, firms can only successfully push wages down if they constrain the amount of labor hired, which forces them to produce less and to increase prices. And thus collusion in the labor market is shown to harm both consumers and workers, suggesting that competition authorities should use their resources to detect and prevent such practices.

Furthermore we have measured the sources responsible for the different levels of economic variables in the non-cooperative and collusive equilibria. Because in the non-cooperative scenario firms only care about their own profits, they do not internalize the cost of stealing consumers and workers from rival companies and so firms raise the wage and reduce the price in a greater extent than they would under collusion. By eliminating the business and labor force stealing effects, the cartel imposes higher prices and lower levels of wages, employment and production, improving thereby the joint profits of the industry.

Under the one input assumption we show further that collusion in the labor market is equivalent to collusion in the final goods market, since both have the same impact on profits and social welfare. But given that competition authorities punish severely any collusive activities in the final goods market while they rarely investigate similar practices in the labor market, there is a clear and strong incentive for firms to jointly fix wages instead of prices, as a mean to conceal their cooperative behavior. This way, firms who collude in the labor market are able to sustain high price levels while, at the same time, they prevent competition authorities from collecting evidence of cooperative price setting (usually in the form of witnesses, signed documents, confessions and legal recordings of meetings and phone conversations). It is therefore imperative that competition authorities look for evidence of agreements to cooperatively set wages and employment levels as well.

Most results in this chapter could be directly extended to the analysis of collusive agreements to fix the prices of any other inputs, as raw materials, machinery or equipments used in the productive process. The reason why we focus on collusion in the labor market is that it appears to be empirically more relevant and to have greater implications on social welfare. In fact, while firms usually have the ability to affect the wages of labor, most of times they have less market power than their suppliers and cannot fix as easily the prices of other inputs (for instance, when buying raw materials, firms usually face an exogenous price depending on the total amount they are willing to buy). Still, our conclusions may be useful to comprehend the effects of collusion between big retailers with substantial market power, as supermarket stores.

In Chapter 3 we introduce new features in our model that are more specific to labor markets, making the analysis more realistic, but considerably more complex.

Chapter 3

Theory of semi-collusion in the labor market

3.1. Introduction

There is nowadays little disagreement about the role of free competition in the efficiency of the markets and social welfare. In the current free market system, a high degree of competition not only guarantees the supply of a great number and variety of goods at low prices, but also the dispersion of economic power among all individuals of society. Indeed, in very competitive markets each individual is able to earn a salary according to his skills and productivity, as well as a “fair” rent for the capital he was able to save along his life. Unfortunately, the ideal concept of perfect competition so often discussed in economic theory is not always present in real industries. Sometimes, when countries are not sufficiently opened to international trade and transport costs are high, markets are simply not large enough to promote competition. Other times, due to large economies of scale or network effects some goods can only be efficiently produced by one firm. That is the case of the natural monopoly. But perhaps most often, the free market system is jeopardized by the cooperative action of some individuals who conspire against society to reduce the level of competition and to get a monopoly rent.

Almost all cartels uncovered so far have been charged of fixing prices or undertaking any other form of collusive deals in the final good market, while cartels who fix wages are hardly ever investigated, probably due to the enormous concern of competition authorities with the welfare of the consumer relatively to the welfare of the worker. We believe, however, that competition authorities should also be responsible for the prosecution of cartels in labor markets, for at least two important reasons. Firstly, in the absence of any other regulatory authority in charge of preserving competition between employers, the protection of the welfare of the worker is an important mean to achieve justice and efficiency, particularly in modern economies where the large part of the population works for somebody else.[5] Secondly, as we have already seen, collusion in the labor market leads to collusion in the final good market and so it has severe impacts on consumer’s welfare as well.

In Chapter 2 it was established that cooperative agreements between firms cause wages, employment levels and quantities transacted to fall and prices of the final good to rise.

[5] Indeed, in United States nine out of ten working individuals are paid employees, whereas only one is self employed (see Hipple, 2010).

Yet those results were obtained under the assumption that firms are able to cooperatively determine the price of all inputs. In opposition, in this chapter we consider a semi-collusive model where firms are only able to undertake cooperative agreements about the price of some inputs used in their production functions.

The economic concept of "semi-collusion" is not new in economic literature and appears to have been used for the first time by Fershtman and Muller (1986), who defined semi-collusive markets as those "*where rivals compete in one variable (or set of variables) and collude in another*". Nonetheless there are some earlier contributions which, despite not using explicitly the term "semi-collusion", end up discussing the same subject.

According to Steen and *Sørgard* (2009), typical models of semi-collusion are composed of a two-stage game, where in the first stage firms set the non-price variables (which are usually more rigid) and in the second stage they choose prices. Then there are two groups of models, those where firms collude on prices and compete on non-price variables and those where the opposite occurs.

In the first group of models, collusion on prices typically leads to tougher competition in the non-price variables and injures both consumers and firms. Some of the earlier examples include Bloch (1932) and Lorange (1973), who respectively describe a German coal industry cartel in the 20s and a cartel of Norwegian cement producers in the 60s, whose members cooperatively fixed prices and total production levels, but competed on productive capacity. Because the market share of every producer was defined as a function of their productive capacity, collusion led to an inefficient over-investment by all firms. The members of price cartels may also compete in other variables, as advertising (Eckard, 1991) and research and development (Brod and Shivakumar, 1999).

In the second group of models, where firms compete on prices and collude on other variables, semi-collusion always improves profits, but may benefit or hurt consumers depending on the particular characteristics of the game. Some examples include semi-collusion on non-price variables as quality (Deltas and Serfes, 2002), advertising (Simbanegavi, 2009) and research and development (d'Aspremont and Jacquemin, 1988). However, in the words of Steen and *Sørgard* (2009):

"*(...) except for collusion on R&D there are few examples in the literature on collusion on non-price variables. As far as we know, there are only a few studies of collusion on advertising and a study of investment in infrastructure in telecom.*"

Our work belongs to this second group of models and studies collusion on wages (which can also be seen as collusion on the productive capacity), contributing to cover what, to our knowledge, constitutes a gap in the extant literature.

To understand the empirical relevance of the semi-collusion hypothesis, recall two examples of cooperative agreements in the labor market that we discussed in Chapter 2. In 1997 fifteen oil companies were sued for exchanging detailed salary information and discussing the budgets for wages paid to managerial, professional and technical employees. And in 2011 several high technology companies like Google, Apple, Adobe, Intel, Intuit and Pixar were accused of undertaking no-solicitation agreements (also known as no-poaching agreements) against their technical engineers. Although in both cases firms succeeded in suppressing the wages paid to highly productive workers with specific technical skills, they were not able to affect the wages of the many non-specialized workers they employ to perform routine activities and less demanding tasks, who are usually hired in larger and more competitive labor markets. Once the two types of workers have some degree of substitutability and can be used in different combinations to achieve the same final output, this clearly suggests that, in many cases, firms are only able to collude about the price of some inputs.

Interestingly, we find in this chapter that the effects of semi-collusion in the labor market do not differ too much from those obtained when firms fix the price of all inputs. Indeed, we show not only that semi-collusion causes the wages and employment of specialized workers to fall, but it also indirectly leads to collusion in the final good market, by creating an incentive for firms to reduce production and to increase prices. The different levels of the economic variables under semi-collusion and competition are the result of the elimination of the business stealing effect and labor force stealing effect, as well as a dynamic effect that is specific to semi-collusive games.

The remainder of the Chapter 3 is organized as follows. In Section 3.2 we present the general formal model consisting in a two-stage game with price and wage competition, which will be used to analyze and compare the non-cooperative equilibrium with the collusive equilibrium. In the first stage firms either decide cooperatively or individually the wages paid to specialized workers and in the second stage firms set simultaneously (without neither cooperate nor communicate) the prices of the final good and the number of non-specialized employees hired. Because the model is solved by backward induction, we begin by solving the second stage in Section 3.3, next we provide the competitive solution for the first stage in Section 3.4 and we describe the collusive solution for the first stage in Section 3.5. In Section 3.6 we compare the results obtained in the two previous sections to identify the impact of semi-collusion on wages, employment, prices and quantities transacted. In Section 3.7 we discuss how results would change if, in the competitive scenario, the control variables were all set simultaneously. Finally, Section 3.8 concludes.

3.2. The model

Consider an industry composed by n firms producing close substitute goods with technologies that combine two types of labor: highly qualified workers (L_{1_i}), paid at the wage rate W_i, and non-specialized workers (L_{2_i}), hired in a perfectly competitive labor market at the wage exogenously fixed at $\bar{W}$. The production function $h_i(L_{1_i}, L_{2_i})$ is concave and increases at decreasing rates with respect to any input. However any increase in one of the inputs raises the marginal productivity of the other. In mathematical notation:

$$\frac{\partial h_i(L_{1_i}, L_{2_i})}{\partial L_{1_i}} > 0, \qquad \frac{\partial h_i(L_{1_i}, L_{2_i})}{\partial L_{2_i}} > 0,$$

$$\frac{\partial^2 h_i(L_{1_i}, L_{2_i})}{\partial {L_{1_i}}^2} \leq 0, \qquad \frac{\partial^2 h_i(L_{1_i}, L_{2_i})}{\partial {L_{2_i}}^2} \leq 0, \qquad \frac{\partial^2 h_i(L_{1_i}, L_{2_i})}{\partial L_{1_i} \partial L_{2_i}} \geq 0.$$

Because the final goods are not perfect substitutes, any firm i of the industry faces a continuous demand function $Q_i = f_i(P_1, \dots, P_n)$ decreasing with respect to its own price and increasing with respect to the prices of the other firms. Similarly, we consider the job posts to be differentiated and hence each firm also faces a continuous specialized labor supply function $L_{1_i} = g_i(W_1, \dots, W_n)$ that rises with W_i and falls with W_j:

$$\frac{\partial f_i(P_1, \dots, P_n)}{\partial P_i} < 0, \qquad \frac{\partial g_i(W_1, \dots, W_n)}{\partial W_i} > 0, \qquad \forall\, i = 1, \dots, n$$

$$\frac{\partial f_i(P_1, \dots, P_n)}{\partial P_j} > 0, \qquad \frac{\partial g_i(W_1, \dots, W_n)}{\partial W_j} < 0, \qquad \forall\, j \neq i.$$

Given the production technologies and the information available about final good demand and labor supply, the firms of the industry repeatedly play a two-stage game in an infinite time horizon, which we will now describe.

In the first stage of the game, all firms set the wage of specialized workers and hire the specialized labor force under two possible equilibrium behaviors. At the non-cooperative equilibrium firms compete with each other and set the wage that optimizes their individual profits, taking the decisions of the other players as given. If, on the other hand, they are able to coordinate their strategies (labor market collusive equilibrium), firms cooperatively set the wages of specialized workers that maximize joint profits. In the latter case, as long as every firm earns a share of the collusion gains and the discount rate is sufficiently close to one, the collusive equilibrium can be sustained with the trigger strategies described in Friedman (1971) or any similar strategies.

In the second stage, firms hire any amount of non-specialized labor they wish at the wage exogenously fixed, set the price of the good produced with the two types of labor and sell it to the final consumer. Naturally they are always subject to the constraint that total sales cannot overcome the total production level. At this stage firms are not able to collude in any dimension.[6]

It is important to briefly discuss why was the model set up in this particular sequence of events, that is, why have we assumed that the wage of specialized workers is determined before the remaining control variables. As it is commonly considered in the economic literature, wages are a relatively rigid variable that cannot be changed very often, especially in the case of high-skilled workers who are the hardest to attract and contract, while prices and employment of non-skilled labor can be more easily adapted to the short run. And so it seems reasonable to consider that once the wages are determined in the industry they cannot be changed again until the next period, while firms can still freely modify prices and employment of non-skilled workers. But if we think about the specific case of the collusive equilibrium, it is easy to understand that this particular sequence of events is, in fact, the only possible solution. Indeed, once the wages are centrally determined by the cartel, they cannot be changed anymore, but firms are still able to take advantage of any unilateral profitable deviations by changing their prices and amounts of non-specialized labor to improve their individual profits. Nevertheless, we cannot absolutely reject the hypothesis that all the decision variables can be set simultaneously in the non-cooperative equilibrium. In Section 3.7 we discuss how the results would be affected in such case.

Next we solve the standard two-stage model following the usual backward induction procedure and thereby we start by determining the equilibrium at the second stage.

[6] The inability of firms to collude in the second stage may result from the absence of market power in the non-specialized labor market and from the actions of competition authority to prevent cooperative price-fixing. Alternatively we can assume that firms are not able to coordinate all decision variables due to asymmetry of information or that they simply prefer to compete in some dimensions.

3.3. Second stage

At the second stage of the game the rational firm chooses the price of the final good and the amount of non specialized labor that optimize its profits, holding the decisions of the other firms fixed. The firm is also constrained by its production technology, as it must acquire the necessary inputs to produce any output sold. At this point the wages of specialized workers can be treated as mere parameters of the model, because they have already been determined in the previous stage and cannot be changed anymore. Then the optimizing problem can be formally expressed as:

$$Max_{P_i, L_{2_i}} \;\; P_i f_i(P_1, \dots, P_n) - W_i g_i(W_1, \dots, W_n) - \bar{W} L_{2_i}$$

$$s.t. \quad f_i(P_1, \dots, P_n) = h_i\big(g_i(W_1, \dots, W_n), L_{2_i}\big).$$

To solve this problem we set up the Lagrangian function and introduce the Lagrange multiplier λ_i, which can be interpreted as the shadow price of the final good.

$$L_{Firm} = P_i f_i(P_1, \dots, P_n) - W_i g_i(W_1, \dots, W_n) - \bar{W} L_{2_i} + \lambda_i\big[h_i\big(g_i(W_i, W_{-i}), L_{2_i}\big) - f_i(P_i, P_{-i})\big].$$

As long as the sufficient conditions hold $\big(\partial^2 f_i(.)/\partial P_i{}^2 \leq 0 \; and \; \partial h_i(.)/\partial L_{i\,2}^2 \leq 0\big)$, the optimization problem is well defined and has an interior solution, which can be found using the following first order conditions:

First order condition with respect to P_i:

$$\frac{\partial L_{Firm}}{\partial P_i} = 0 \quad \Leftrightarrow \quad f_i(.) + P_i \frac{\partial f_i(.)}{\partial P_i} - \lambda_i \frac{\partial f_i(.)}{\partial P_i} = 0 \quad \Leftrightarrow$$

$$\Leftrightarrow \quad f_i(.) + (P_i - \lambda_i) \frac{\partial f_i(.)}{\partial P_i} = 0. \tag{3.3.1}$$

First order conditions with respect to L_{2_i}:

$$\frac{\partial L_{Firm}}{\partial L_{2_i}} = 0 \quad \Leftrightarrow \quad -\bar{W} + \lambda_i \frac{\partial h_i(.)}{\partial L_{2_i}} = 0 \quad \Leftrightarrow \quad \lambda_i = \bar{W} \frac{\partial L_{2_i}}{\partial h_i}. \tag{3.3.2}$$

First order conditions with respect to λ_i:

$$\frac{\partial L_{Firm}}{\partial \lambda_i} = 0 \quad \Leftrightarrow \quad f_i(.) = h_i\big(g_i(.), L_{2_i}\big). \tag{3.3.3}$$

Because our variables of interest are L_{2_i} and P_i, equation (3.3.2) can be used to eliminate the shadow price from equation (3.3.1), so that first order conditions are rewritten as:

$$f_i(P_1, \dots, P_n) = h_i\big(g_i(W_1, \dots, W_n), L_{2_i}\big). \tag{3.3.4}$$

$$f_i(P_1, \dots, P_n) + \left(P_i - \bar{W}\frac{\partial L_{2_i}}{\partial h_i}\right)\frac{\partial f_i(P_1, \dots, P_n)}{\partial P_i} = 0. \tag{3.3.5}$$

Equation (3.3.4) is simply the production technology constraint. Equation (3.3.5) states that under optimal behavior the marginal impact on profits of a slight price change must be null. In other words, when an optimizing firm increases the price in one unit, the extra unit of money it earns for each unit of product sold must exactly offset the loss of sales resulting from the fall in demand.

Applying equations (3.3.4) and (3.3.5) to every firm gives the equilibrium prices and employment levels of non-specialized labor as a function of the wages of specialized workers, $P_i(W_1, \dots, W_n)$and $L_{2_i}(W_1, \dots, W_n)$.

3.4. Non-cooperative outcome in the first stage of the game

Now that we have predicted how the firms of the industry will behave in the second stage of the game for given wage levels, we have the necessary information to determine the solution in the first stage, which will crucially depend on whether the firms act non-cooperatively or successfully coordinate strategies. We begin by studying the non-cooperative solution.

As usual, in the non-cooperative equilibrium any firm of the industry chooses the wage level that optimizes its profits holding fixed the decisions of the other players. But since a rational firm is able to predict the behavior of the industry in the second stage, it will not only measure the direct impact of the wage on profits through total costs and production capacity, but it will also account for the repercussions of the wage on future decisions about prices and employment levels of non-specialized workers, which affect profits as well. The maximizing problem of the firm can be therefore formally represented as:

$$Max_{W_i} \;\; P_i(W_1,\dots,W_n) f_i\big(P_1(W_1,\dots,W_n),\dots,P_n(W_1,\dots,W_n)\big) - W_i g_i(W_1,\dots,W_n) - \bar{W} L_{2_i}(W_1,\dots,W_n)$$

$$s.t. \quad f_i\big(P_1(W_1,\dots,W_n),\dots,P_n(W_1,\dots,W_n)\big) = h_i\big(g_i(W_1,\dots,W_n), L_{2_i}(W_1,\dots,W_n)\big).$$

As we can see, the prices and employment levels of non-specialized workers appear in the expression as a function of wages. To solve this problem we set up again the Lagrange function, whose Lagrange multiplier is also a function of wages:

$$L_i = P_i(W_1,\dots,W_n) f_i\big(P_1(W_1,\dots,W_n),\dots,P_n(W_1,\dots,W_n)\big) -$$

$$-W_i g_i(W_1,\dots,W_n) - \bar{W} L_{2_i}(W_1,\dots,W_n) +$$

$$+\lambda_i(W_1,\dots,W_n)\left[h_i\left(g_i(W_1,\dots,W_n), L_{2_i}(W_1,\dots,W_n)\right) - f_i\big(P_1(W_1,\dots,W_n),\dots,P_n(W_1,\dots,W_n)\big)\right].$$

Given the assumptions we have made about the product demand, labor supply and production functions, the Lagrangian is concave and has an interior optimal solution at the point where its derivative with respect to the wage is equal to zero:

$$\frac{\partial L_i}{\partial W_i} = \frac{\partial P_i(.)}{\partial W_i} f_i(.) + \ P_i(.) \sum_{k=1}^{n} \left\{ \frac{\partial f_i(.)}{\partial P_k} \frac{\partial P_k(.)}{\partial W_i} \right\} -$$

$$-g_i(.) - W_i \frac{\partial g_i(.)}{\partial W_i} - \overline{W} \frac{\partial L_{2i}(.)}{\partial W_i} + \frac{\partial \lambda_i(.)}{\partial W_i} [h_i(.) - f_i(.)] +$$

$$+\lambda_i(.) \left[\frac{\partial h_i(.)}{\partial g_i} \frac{\partial g_i(.)}{\partial W_i} + \frac{\partial h_i(.)}{\partial L_{2i}} \frac{\partial L_{2i}(.)}{\partial W_i} - \sum_{k=1}^{n} \left\{ \frac{\partial f_i(.)}{\partial P_k} \frac{\partial P_k(.)}{\partial W_i} \right\} \right] = 0.$$

Splitting and rearranging the components of the last term allows us to rewrite the first order condition as:

$$\frac{\partial P_i(.)}{\partial W_i} f_i(.) + \ \left(P_i(.) - \lambda_i(.)\right) \sum_{k=1}^{n} \left\{ \frac{\partial f_i(.)}{\partial P_k} \frac{\partial P_k(.)}{\partial W_i} \right\} -$$

$$-g_i(.) + \left(\lambda_i(.) \frac{\partial h_i(.)}{\partial g_i} - W_i \right) \frac{\partial g_i(.)}{\partial W_i} + \left(\lambda_i(.) \frac{\partial h_i(.)}{\partial L_{2i}} - \overline{W} \right) \frac{\partial L_{2i}(.)}{\partial W_i} +$$

$$+ \frac{\partial \lambda_i(.)}{\partial W_i} [h_i(.) - f_i(.)] = 0.$$

The last expression can be further simplified using the first order conditions previously obtained for P_i, L_{2i} and $\lambda_i(.)$. From equation (3.3.4) we know that the production level must met the quantity demanded for the final good, that is, $h_i(.) = f_i(.)$. Equation (3.3.2) can be used to express the shadow price as the cost of producing one unit of the final good using non-specialized labor, $\lambda_i(.) = \overline{W} \frac{\partial L_{2i}(.)}{\partial h_i(.)}$. Replacing these two results in the previous equation:

$$\frac{\partial P_i(.)}{\partial W_i} f_i(.) + \ \left(P_i(.) - \overline{W} \frac{\partial L_{2i}(.)}{\partial h_i} \right) \sum_{k=1}^{n} \left\{ \frac{\partial f_i(.)}{\partial P_k} \frac{\partial P_k(.)}{\partial W_i} \right\} -$$

$$-g_i(.) + \left(\overline{W} \frac{\partial L_{2i}(.)}{\partial h_i} \frac{\partial h_i(.)}{\partial g_i} - W_i \right) \frac{\partial g_i(.)}{\partial W_i} = 0.$$

Breaking the terms inside the sum gives:

$$\frac{\partial P_i(.)}{\partial W_i} \left[f_i(.) + \left(P_i(.) - \overline{W} \frac{\partial L_{2i}(.)}{\partial h_i} \right) \frac{\partial f_i(.)}{\partial P_i} \right] +$$

$$+\left(P_i(.) - \bar{W}\frac{\partial L_{2i}(.)}{\partial h_i}\right)\sum_{k\neq i}^{n-1}\left\{\frac{\partial f_i(.)}{\partial P_k}\frac{\partial P_k(.)}{\partial W_i}\right\} -$$

$$-g_i(.) + \left(\bar{W}\frac{\partial L_{2i}(.)}{\partial h_i}\frac{\partial h_i(.)}{\partial g_i} - W_i\right)\frac{\partial g_i(.)}{\partial W_i} = 0.$$

Finally, equation (3.3.5) can be used to replace $f_i(.) + \left(P_i(.) - \bar{W}\frac{\partial L_{2i}(.)}{\partial h_i}\right)\frac{\partial f_i(.)}{\partial P_i}$ by zero, allowing us to rewrite the first order condition with respect to the specialized labor wage as:

$$-g_i(.) + \left(\bar{W}\frac{\partial L_{2i}(.)}{\partial h_i}\frac{\partial h_i(.)}{\partial g_i} - W_i\right)\frac{\partial g_i(.)}{\partial W_i} +$$

$$+\left(P_i(.) - \bar{W}\frac{\partial L_{2i}(.)}{\partial h_i}\right)\sum_{k\neq i}^{n-1}\left\{\frac{\partial f_i(.)}{\partial P_k}\frac{\partial P_k(.)}{\partial W_i}\right\} = 0. \qquad (3.4.1)$$

As one would expect, equation (3.4.1) states that at the non-cooperative equilibrium no profitable deviations exist, that is, a small variation in the wage does not affect the profits of the firm. In the left-handed side of equation (3.4.1) we observe the three different effects of raising the wage of specialized workers in one unit: firstly, the firm loses one unit of money for each worker employed (first term of equation (3.4.1)); secondly, the firm is able to substitute some of the non-specialized labor force by specialized labor, saving thus in the wages paid to low-skilled employees (second term of equation (3.4.1)); and thirdly the remaining firms of the industry will react to the new wage with different price levels, which in turn will affect the total amount of the final good that firm i is able to sell (third term of equation (3.4.1)).

The last effect of augmenting the wage is probably the most interesting and unexpected, since it results from the specific dynamics of this game. Its sign depends on how the equilibrium prices in the second stage react to a change in the wage level, $\partial P_k(.)/\partial W_i, k \neq i$.

In Appendix C we perform a comparative-static analysis to prove that the marginal effect of the wage of firm i on its own price is negative, $\partial P_i(.)/\partial W_i < 0$. Intuitively a firm that has already committed to a higher wage and employed a greater specialized labor force in the first stage has an incentive to increase the production level in the second stage, which can only be sold at a lower price. Then we show in Appendix C that the marginal effect of the wage of firm i on the prices of the other firms, $\partial P_k(.)/\partial W_i$, can be either positive or negative, depending on the specific characteristics of the industry. In fact, the sign of $\partial P_k(.)/\partial W_i$ relies on the dimension of two distinct forces operating in different directions: on the one hand, when firm i increases the wage, it gives a clear sign that it will reduce prices in the second stage and so the rest of the industry is likely to respond with lower prices as well; on the other hand, all other firms are now able to hire less specialized workers for the same wage levels and face a greater marginal cost of production, which induces them to raise prices.

We classify industries in two different types according to the sign of $\partial P_k(.)/\partial W_i$. In *type A* industries, job posts are well differentiated and final goods are very close substitutes (and so demand is very reactive to the prices of the alternative goods). As a result, the first force dominates the second and $\partial P_k(.)/\partial W_i$ is negative. In this case firms are usually more reluctant to increase wages as they do not want to trigger a price war – ***price war effect***.

In *type B* industries, because final goods are sufficiently different at the eyes of consumers and post jobs are close substitutes, the second force dominates the first and $\partial P_k(.)/\partial W_i$ is positive. When this occurs, firms have an extra temptation to raise the wage as a strategic move to undermine the production capacity of the remaining firms, forcing them to produce less and to charge higher prices. We call this move a ***shooting the moon strategy***, since it is extremely sophisticated and risky to increase the wage not to attract additional workers, but to steal business from the rival companies of the industry without having to decrease the price. Such strategy can easily backfire when the whole industry attempts to *shoot the moon*, case in which every firm ends paying a greater salary without being able to steal any sales from other companies.

As we will see later, the sign of $\partial P_k(.)/\partial W_i$ has an important role in the results of our thesis.

3.5. Collusive outcome in the first stage of the game

To find the collusive equilibrium in the first stage of the game, we use similar procedures to those of the last section, except that the wage of specialized workers paid by a particular firm is now centrally determined to maximize the joint profits of the cartel. Here is the new optimization problem:

$$Max_{W_i} \quad \sum_{j=1}^{n} P_j(W_1, \dots, W_n) f_j\big(P_1(W_1, \dots, W_n), \dots, P_n(W_1, \dots, W_n)\big)$$

$$- \sum_{j=1}^{n} W_j g_j(W_1, \dots, W_n) - \sum_{j=1}^{n} \overline{W} L_{2_j}(W_1, \dots, W_n)$$

$s.t.$

$$\begin{cases} f_1\big(P_1(W_1, \dots, W_n), \dots, P_n(W_1, \dots, W_n)\big) = h_1\big(g_1(W_1, \dots, W_n), L_{2_1}(W_1, \dots, W_n)\big) \\ \dots \\ f_n\big(P_1(W_1, \dots, W_n), \dots, P_n(W_1, \dots, W_n)\big) = h_n\big(g_n(W_1, \dots, W_n), L_{2_n}(W_1, \dots, W_n)\big) \end{cases}.$$

Note that, just as before, the wage is determined taking into account the impact it has on the future decisions of firms regarding prices and non-specialized labor hired. To solve the maximization problem we set the Lagrangian of the cartel.

$$L = \sum_{j=1}^{n} P_j(W_1, \dots, W_n) f_j\big(P_1(W_1, \dots, W_n), \dots, P_n(W_1, \dots, W_n)\big) -$$

$$- \sum_{j=1}^{n} W_j g_j(W_1, \dots, W_n) - \sum_{j=1}^{n} \overline{W} L_{2_j}(W_1, \dots, W_n) +$$

$$+ \sum_{j=1}^{n} \lambda_j\,(W_1, \dots, W_n) \Big[h_j\Big(g_j(W_1, \dots, W_n), L_{2_j}(W_1, \dots, W_n)\Big)$$

$$- f_j\big(P_1(W_1, \dots, W_n), \dots, P_n(W_1, \dots, W_n)\big)\Big].$$

First order conditions require that at the interior optimal solution a small variation of the wage neither raises nor reduces the joint profits of the industry:

$$\frac{\partial L}{\partial W_i}=\sum_{j=1}^{n}\left\{\frac{\partial P_j(.)}{\partial W_i}f_j(.)\right\}+\sum_{j=1}^{n}\left\{\left(P_j(.)-\lambda_j(.)\right)\sum_{k=1}^{n}\frac{\partial f_j(.)}{\partial P_k}\frac{\partial P_k(.)}{\partial W_i}\right\}-$$

$$-g_i(.)+\sum_{j=1}^{n}\left\{\left(\lambda_j(.)\frac{\partial h_j(.)}{\partial g_j}-W_j\right)\frac{\partial g_j(.)}{\partial W_i}\right\}+$$

$$+\sum_{j=1}^{n}\left\{\left(\lambda_j(.)\frac{\partial h_j(.)}{\partial L_{2j}}-\bar{W}\right)\frac{\partial L_{2j}(.)}{\partial W_i}\right\}+\sum_{j=1}^{n}\left\{\frac{\partial \lambda_j(.)}{\partial W_i}\left[h_j(.)-f_j(.)\right]\right\}=0.$$

Once again, the expression above can be simplified with the first order conditions obtained in the second stage of the game. Using equation (3.3.2) to get rid of the Lagrangian multiplier and equation (3.3.4) to eliminate the last term:

$$\sum_{j=1}^{n}\left\{\frac{\partial P_j(.)}{\partial W_i}f_j(.)\right\}+\sum_{j=1}^{n}\left\{\left(P_j(.)-\bar{W}\frac{\partial L_{2j}(.)}{\partial h_j}\right)\sum_{k=1}^{n}\frac{\partial f_j(.)}{\partial P_k}\frac{\partial P_k(.)}{\partial W_i}\right\}-$$

$$-g_i(.)+\sum_{j=1}^{n}\left\{\left(\bar{W}\frac{\partial L_{2j}(.)}{\partial h_j}\frac{\partial h_j(.)}{\partial g_j}-W_j\right)\frac{\partial g_j(.)}{\partial W_i}\right\}=0.$$

Splitting the term $\sum_{k=1}^{n}\frac{\partial f_j(.)}{\partial P_k}\frac{\partial P_k(.)}{\partial W_i}$ gives:

$$\sum_{j=1}^{n}\left\{\frac{\partial P_j(.)}{\partial W_i}\left[f_j(.)+\left(P_j(.)-\bar{W}\frac{\partial L_{2j}(.)}{\partial h_j}\right)\frac{\partial f_j(.)}{\partial P_j}\right]\right\}+$$

$$+\sum_{j=1}^{n}\left\{\left(P_j(.)-\bar{W}\frac{\partial L_{2j}(.)}{\partial h_j}\right)\sum_{k\neq j}^{n-1}\frac{\partial f_j(.)}{\partial P_k}\frac{\partial P_k(.)}{\partial W_i}\right\}-$$

$$-g_i(.)+\sum_{j=1}^{n}\left\{\left(\bar{W}\frac{\partial L_{2j}(.)}{\partial h_j}\frac{\partial h_j(.)}{\partial g_j}-W_j\right)\frac{\partial g_j(.)}{\partial W_i}\right\}=0.$$

At last, from equation (3.3.5) we know that $f_j(.)+\left(P_j(.)-\bar{W}\frac{\partial L_{2j}(.)}{\partial h_j}\right)\frac{\partial f_j(.)}{\partial P_j}=0$ and so the final expression for the first order condition with respect to the wage of highly skilled workers is:

$$-g_i(.)+\sum_{j=1}^{n}\left\{\left(\bar{W}\frac{\partial L_{2j}(.)}{\partial h_j}\frac{\partial h_j(.)}{\partial g_j}-W_j\right)\frac{\partial g_j(.)}{\partial W_i}\right\}+$$

$$+\sum_{j=1}^{n}\left\{\left(P_j(.)-\bar{W}\frac{\partial L_{2j}(.)}{\partial h_j}\right)\sum_{k\neq j}^{n-1}\frac{\partial f_j(.)}{\partial P_k}\frac{\partial P_k(.)}{\partial W_i}\right\}=0. \qquad (3.5.1)$$

In conclusion, when the firms of the industry form a cartel in the labor market, the wage paid by a particular firm is established to optimize the joint profits of the whole industry, which are affected through three different channels. Firstly, when the wage of firm i increases in one unit, the cartel loses one unit of money for each worker employed at that firm. Secondly, an increase in the wage of firm i has a positive effect on the specialized labor force of that firm, but a negative effect on the other companies, who are compelled to hire more unskilled workers. And finally, a change in the wage is replied with changes in prices and quantities, affecting the sales revenues of every firm.

3.6. Non-cooperative equilibrium vs. collusive equilibrium

Once it has been determined how the firms in the industry strategically interact under competitive and collusive settings, we can perform a comparative static analysis between the two equilibria to predict the effects of collusion in the labor market on the main economic variables. Starting by the impact of collusion on the wage of specialized workers, we use the first order conditions in equations (3.4.1) and (3.5.1) to compute the difference between the marginal effect of the wage on the joint profits of the industry and on the profit of the firm:

$$\frac{\partial L}{\partial W_i} - \frac{\partial L_i}{\partial W_i} = \sum_{j \neq i}^{n-1} \left\{ \left(P_j(.) - \bar{W} \frac{\partial L_{2j}(.)}{\partial h_j} \right) \frac{\partial f_j(.)}{\partial P_i} \frac{\partial P_i(.)}{\partial W_i} \right\}$$
$$+ \sum_{j \neq i}^{n-1} \left\{ \left(\bar{W} \frac{\partial L_{2j}(.)}{\partial h_j} \frac{\partial h_j(.)}{\partial g_j} - W_j \right) \frac{\partial g_j(.)}{\partial W_i} \right\}$$
$$+ \sum_{j \neq i}^{n-1} \left\{ \left(P_j(.) - \bar{W} \frac{\partial L_{2j}(.)}{\partial h_j} \right) \sum_{\substack{k \neq j \\ k \neq i}}^{n-2} \frac{\partial f_j(.)}{\partial P_k} \frac{\partial P_k(.)}{\partial W_i} \right\}. \qquad (3.6.1)$$

As long as that difference is negative, a positive variation in the wage paid by firm i to specialized workers has a greater impact on its own individual profits than on the profits of the whole cartel $\left(\frac{\partial L_i}{\partial W_i} > \frac{\partial L}{\partial W_i} \right)$, meaning that under competition any firm is willing to set a higher wage and to employ more specialized workers. To prove that the difference between the two marginal effects is, indeed, negative, we must analyze the meaning and sign of each of the three components in the right-handed side of equation (3.6.1). For now we consider only *type A* industries where $\frac{\partial P_k(.)}{\partial W_i}$ is negative.

The first component of equation (3.6.1), $\sum_{j \neq i}^{n-1} \left\{ \left(P_j(.) - \bar{W} \frac{\partial L_{2j}(.)}{\partial h_j} \right) \frac{\partial f_j(.)}{\partial P_i} \frac{\partial P_i(.)}{\partial W_i} \right\}$, can be interpreted as the ***business stealing effect***: when a firm in the industry increases the wage paid to specialized workers and expands its productive capacity, it is forced to reduce the price of the final good to meet the new production levels. And, as a result, the firm ends up stealing indirectly some sales from the rival companies. At the collusive equilibrium firms refrain from hurting each other and eliminate the business stealing effect.

As regards to the sign of this component, the term $\frac{\partial P_i(.)}{\partial W_i}$ is proved to be negative in Appendix C, $\frac{\partial f_j(.)}{\partial P_i}$ is positive by definition and $P_j(.) - \bar{W}\,\partial L_{2j}(.)/\partial h_j$ corresponds to the mark-up of the final good (price minus the marginal cost expressed in terms of non-specialized labor), which is clearly positive for any j or firms would be producing unprofitable units of product. Indeed, from equation (3.3.5):

$$f_j(.) + \left(P_j - \bar{W}\frac{\partial L_{2j}}{\partial h_j}\right)\frac{\partial f_j(.)}{\partial P_j} = 0 \quad \Leftrightarrow \quad P_j - \bar{W}\frac{\partial L_{2j}}{\partial h_j} = -\frac{f_j(.)}{\partial f_j(.)/\partial P_j} \quad \Leftrightarrow$$

$$P_j - \bar{W}\frac{\partial L_{2j}}{\partial h_j} > 0.$$

Because the product of the three terms in the first component of equation (3.6.1) is negative, the elimination of the *business stealing effect* leads to lower wages under collusion.

The second component of equation (3.6.1), $\sum_{j\neq i}^{n-1}\left\{\left(\bar{W}\frac{\partial L_{2j}(.)}{\partial h_j}\frac{\partial h_j(.)}{\partial g_j} - W_j\right)\frac{\partial g_j(.)}{\partial W_i}\right\}$, corresponds to the ***(specialized) labor force stealing effect***: when the same firm of the industry raises the wage to its individually optimal level, it steals specialized workers from the other firms, who incur in extra costs to use non skilled labor instead. Under collusion the labor force stealing effect is eliminated, since firms account for the loss imposed on the rest of the industry when they attempt to steal workers from each other.

While the sign of $\partial g_j(.)/\partial W_i$ is negative by definition, the term $\bar{W}\frac{\partial L_{2j}(.)}{\partial h_j}\frac{\partial h_j(.)}{\partial g_j} - W_j$ (which can be interpreted as the additional cost of using non specialized labor to make the job of one highly skilled worker) is always positive in *type A* industries, or firms would prefer to use only non specialized workers to produce the whole output. In fact, rearranging equation (3.4.1) gives:

$$\left(\bar{W}\frac{\partial L_{2i}(.)}{\partial h_i}\frac{\partial h_i(.)}{\partial g_i} - W_i\right)\frac{\partial g_i(.)}{\partial W_i} = g_i(.) - \left(P_i(.) - \bar{W}\frac{\partial L_{2i}(.)}{\partial h_i}\right)\sum_{k\neq i}^{n-1}\left\{\frac{\partial f_i(.)}{\partial P_k}\frac{\partial P_k(.)}{\partial W_i}\right\}.$$

As we can see above, for $\partial P_k(.)/\partial W_i < 0$ the right-handed side of the last equation is positive and so the term $\bar{W}\frac{\partial L_{2i}(.)}{\partial h_i}\frac{\partial h_i(.)}{\partial g_i} - W_i$ in the left-handed side must be positive as well. Because the product of the two terms of the labor force stealing effect is negative, its elimination also leads to lower wages under collusion.

The final component of equation (3.6.1), $\sum_{j\neq i}^{n-1}\left\{\left(P_j(.) - \bar{W}\frac{\partial L_{2j}(.)}{\partial h_j}\right)\sum_{\substack{k\neq j\\k\neq i}}^{n-2}\frac{\partial f_j(.)}{\partial P_k}\frac{\partial P_k(.)}{\partial W_i}\right\}$, corresponds to an intensification of the ***price war effect***. As we have already seen, at the non-cooperative equilibrium firms in *type A* industries avoid raising wages too much, as they do not want to trigger a price war in the second stage. But under collusion, an optimizing firm does not only internalize the costs of a price war on its own profits, but on the profitability of the whole industry. As a result, at the cooperative equilibrium firms decrease wages paid to specialized workers even more due to an amplification of the price war effect.

It is easy to show that the sign of this component is also negative, since the mark-up $\left(P_j(.) - \bar{W}\frac{\partial L_{2j}(.)}{\partial h_j}\right)$ is positive and the term $\sum_{\substack{k\neq j\\k\neq i}}^{n-2}\frac{\partial f_j(.)}{\partial P_k}\frac{\partial P_k(.)}{\partial W_i}$ is negative in *type A* industries.

Given the negative signs of the three different components in the right-handed side of equation (3.6.1) we verify that $\frac{\partial L}{\partial W_i} - \frac{\partial L_i}{\partial W_i} < 0$ and hence the wages paid to non specialized workers are lower under collusion. The difference between the wage levels in the two distinct scenarios is the result of the elimination of the business stealing effect, the labor force stealing effect and the amplification of the price war effect.

Interestingly things are slightly different as far as *type B* industries are concerned, where $\partial P_k(.)/\partial W_i$ is positive. For those firms, ***shooting the moon strategies*** are so appealing that at the non-cooperative equilibrium firms commit to very high wages and overly employ specialized workers in order to enforce the other companies to practice high prices, even though it would be cheaper to produce the same output level with more intensive combinations of non skilled labor. As we prove below, for $\partial P_k(.)/\partial W_i$ large enough, the term $\bar{W}\frac{\partial L_{2i}(.)}{\partial h_i}\frac{\partial h_i(.)}{\partial g_i} - W_i$ is negative and the second component in equation (3.6.1), which represents the elimination of the labor force stealing effect, becomes positive. This means that under collusion firms get an incentive to augment wages in order to relief the other firms from the excessive specialized labor force they have. From equation (3.4.1):

$$\left(\bar{W}\frac{\partial L_{2_i}(.)}{\partial h_i}\frac{\partial h_i(.)}{\partial g_i}-W_i\right)\frac{\partial g_i(.)}{\partial W_i}=g_i(.)-\left(P_i(.)-\bar{W}\frac{\partial L_{2_i}(.)}{\partial h_i}\right)\sum_{k\neq i}^{n-1}\left\{\frac{\partial f_i(.)}{\partial P_k}\frac{\partial P_k(.)}{\partial W_i}\right\}$$

$$\Rightarrow\quad\left(\bar{W}\frac{\partial L_{2_i}(.)}{\partial h_i}\frac{\partial h_i(.)}{\partial g_i}-W_i\right)<0,\qquad for\ \partial P_k(.)/\partial W_i\ large\ enough.$$

In addition, in *type B* industries where $\partial P_k(.)/\partial W_i$ is positive, the third component of equation (3.6.1) also becomes positive and can be now interpreted as an intensification of the ***shooting the moon strategies***. In this type of industries, an optimizing firm operating in a cartel has an extra incentive to keep high wages, as it accounts for a positive side effect of the shooting the moon strategy on the profits of the other firms, who are able to increase their sales due to the higher prices practiced in the rest of the industry.

Despite the positive signs of the labor force stealing effect and shooting the moon strategies in *type B* industries, we show in Appendix F that the business stealing effect component remains negative and offsets the other two components. As a result the wages paid to specialized workers prevail lower under collusion.

With regard to the impact of collusion on the other economic variables, we compute their average behavior using symmetry.

Given that the firms of a cartel pay on average lower wages, the effect of collusion on the employment of specialized workers can be directly inferred from the labor supply functions. Under symmetry,

$$L_{1_i}=g_i(W_1,\dots,W_n)=\frac{G(W_1,\dots,W_n)}{n}.$$

As the total specialized labor supply $G(W_1,\dots,W_n)$ is positively correlated with every W, when firms undertake cooperative agreements to reduce wages they are able to hire fewer specialized workers.

Concerning the prices of the final good, it has already been discussed how the firms of the industry react when one particular firm decreases the wage. Now we are interested in predicting the reaction of the industry when all firms decrease the wage or, similarly, when they all hire less specialized workers. In Appendix G we show that as a result of their smaller productive capacity their best reply is to charge a higher price. And given the market demand functions and the symmetry assumption, this means that firms are able to sell less:

$$Q_i = f_i(P_1, \dots, P_n) = \frac{F(P_1, \dots, P_n)}{n}.$$

Ultimately, the impact of collusion in the employment level of non specialized labor is undetermined. The ambiguity of the sign results from two opposing forces. When the industry forms a cartel and sets lower wages for specialized workers, firms are able to attract a smaller amount of highly skilled labor and, for the same production level, they need more non-specialized workers than before (substitution effect). On the other hand when firms set lower wages they have an incentive to produce less in the second stage of the game and thus to hire fewer non-specialized workers as well, whose marginal productivity is now smaller (scale effect).

3.7. Simultaneous competitive game

We have asserted that the sequential order of decisions in the two stage game previously described is the only adequate way to model the semi-collusive interaction between firms who cooperatively fix the wages of specialized workers. After all, if the prices and employment levels of non-specialized labor were non-cooperatively defined firstly, once the wages were centrally determined firms would choose new prices and employment levels to optimize their individual profits. The sequentiality of the two stage model also seems acceptable in the free competition scenario, because wages of specialized labor are often a variable more rigid than prices and low-skilled workers hired. Notwithstanding we cannot discard the hypothesis that under free competition all variables are set simultaneously. In that case, despite the first order conditions given in equations (3.3.4) and (3.3.5) remaining the same, the wage of specialized workers is now set to optimize individual profits without considering any future impact on prices and non skilled labor. That is, wages are set to solve the following optimization problem:

$$Max_{W_i,P_i,L_{2i}} Profit_i \ = P_i f_i(P_1,\dots,P_n) - W_i g_i(W_1,\dots,W_n) - \overline{W} L_{2_i}$$

$$s.t. \quad f_i(P_1,\dots,P_n) = h_i\big(g_i(W_1,\dots,W_n), L_{2_i}\big).$$

The Lagrangian function of the firm becomes

$$L_{Firm} = P_i f_i(P_1,\dots,P_n) - W_i g_i(W_1,\dots,W_n) - \overline{W} L_{2_i} + \lambda_i\big[h_i\big(g_i(W_i,W_{-i}), L_{2_i}\big) - f_i(P_i,P_{-i})\big].$$

and the first order condition is:

$$\frac{\partial L_{Firm}}{\partial W_i} = 0 \quad \Leftrightarrow \quad -g_i(.) - W_i \frac{\partial g_i(.)}{\partial W_i} + \lambda_i \frac{\partial h_i(.)}{\partial g_i} \frac{\partial g_i(.)}{\partial W_i} = 0 \quad \Leftrightarrow$$

$$\Leftrightarrow \quad -g_i(.) + \left(\lambda_i \frac{\partial h_i(.)}{\partial g_i} - W_i\right) \frac{\partial g_i(.)}{\partial W_i} = 0.$$

Substituting λ_i by equation (3.3.2):

$$-g_i(.) + \left(\overline{W} \frac{\partial L_{2_i}(.)}{\partial h_i} \frac{\partial h_i(.)}{\partial g_i} - W_i\right) \frac{\partial g_i(.)}{\partial W_i} = 0. \qquad (3.7.1)$$

The only difference between equations (3.4.1) and (3.7.1) is the absence in the last of the dynamic effect of wages on future decisions about prices. In fact, because all variables are now set simultaneously, the price war or shooting the moon effect disappears. Howsoever, when the cartel is formed, the artificial rigidity imposed on wages reintroduces the dynamic effect and firms become capable of affecting prices and employment levels of non specialized workers through changes in wages of specialized workers. For that reason, the difference between the marginal effect of the wage on the profits of the industry and on the profit of the firm has now a slightly different expression:

$$\begin{aligned}\frac{\partial L}{\partial W_i}-\frac{\partial L_i}{\partial W_i}= &\sum_{j\neq i}^{n-1}\left\{\left(P_j(.)-\bar{W}\frac{\partial L_{2j}(.)}{\partial h_j}\right)\frac{\partial f_j(.)}{\partial P_i}\frac{\partial P_i(.)}{\partial W_i}\right\}\\ &+\sum_{j\neq i}^{n-1}\left\{\left(\bar{W}\frac{\partial L_{2j}(.)}{\partial h_j}\frac{\partial h_j(.)}{\partial g_j}-W_j\right)\frac{\partial g_j(.)}{\partial W_i}\right\}\\ &+\sum_{j=1}^{n}\left\{\left(P_j(.)-\bar{W}\frac{\partial L_{2j}(.)}{\partial h_j}\right)\sum_{\substack{k\neq j\\ k\neq i}}^{n-2}\frac{\partial f_j(.)}{\partial P_k}\frac{\partial P_k(.)}{\partial W_i}\right\}\end{aligned} \quad (3.7.2)$$

We observe in equation (3.7.2) that the formation of a cartel still eliminates the business stealing effect and the (specialized) labor stealing effect, but now it also creates the dynamic mechanism through which firms can use wages to influence the prices of the industry and total sales. Thus the third component of the right-handed side of equation (3.7.2) represents not a mere intensification of effects, but the creation of the price war effect or shooting the moon effect, which did not exist at the simultaneous non-cooperative equilibrium.

While it is easy to see that the three components of equation (3.7.2) are negative in *type A* industries, in Appendix H we prove that $\frac{\partial L}{\partial W_i}-\frac{\partial L_i}{\partial W_i}$ remains negative in *type B* industries, so that wages are, as always, lower under collusion.

3.8. Conclusions

Chapter 3 investigates the relation between semi-collusion in the labor market and collusion in the final good market. When we came up with the idea of addressing this subject for the first time, we were immediately suggested by conventional wisdom that firms who cooperatively fix lower wages have access to cheaper cost technologies and are able, thereby, to produce more at lower prices. As we have already shown in Chapter 2, this is of course not the case under one input production functions, where there is a perfect and direct relation between the two types of collusion: any central decisions to decrease wages reduces the number of workers that firms are able to hire and their productive capacity, forcing them to raise prices. In Chapter 3 we attempted to break the direct relation between collusion in the labor market and collusion in the final good market by allowing firms to use a second input of production, non-specialized workers, over whom they have absolutely no power to affect wages. Nevertheless, the qualitative results remained surprisingly similar to the previous ones, although they were now obtained under a much more general and realistic setting where the interaction between players became substantially complex.

Indeed we have shown in a two-stage game model with two-input production functions that when firms interact cooperatively in the labor market, their only mechanism available to reduce wages is to accept lower levels of employment of specialized workers, which constrains their productive capacity. Even though firms can replace some of the specialized labor by unskilled workers, it is still in their own interest to produce less and to charge higher prices. And so collusion in the labor market leads, as always, to collusion in the final good market.

In the absence of more specific assumptions about demand, supply and production functions, it is not possible to determine in what direction collusion affects the employment of non-specialized labor. One the one hand, at the collusive equilibrium companies hire less specialized workers and thus they must use additional low-skilled employees as an alternative input to obtain the same output level (substitution effect). On the other hand, because at the collusive equilibrium production levels are smaller, firms employ less of every input, including non-specialized labor (scale effect).

The economic motivation for the cartel to coordinate lower wages and higher prices depends on the particular characteristics of the industry. We consider two types of industries which we discuss in turn.

In *type A* industries, where job posts are very differentiated and final goods are close substitutes, the low wage levels fixed under collusion result from the ability of the cartel to eliminate the business stealing effect and the labor force stealing effect and to amplify the price war effect. In other words, the cartel eliminates the individual temptation of firms to steal market quotas and specialized workers from each other and prevents them from triggering price wars.

In *type B* industries, where jobs posts are very close substitutes and final goods are sufficiently differentiated, wages are still lower under collusion due to the elimination of the business stealing effect, even though the cartel rationally prevents wages from falling too much, as it accounts for the positive side effects of the labor force stealing effect and shooting the moon strategies.

In conclusion, we hope to contribute with this work to an evolution of economic policy towards a greater protection of the consumer and the worker. We believe the application of our theory can actually improve the level of competition in markets, which is so essential to the efficient operation of the free market system.

Chapter 4

Collusive Scene Investigation

A modern empirical approach to uncover cartels

4.1. Introduction

As a result of the actions of cartels, millions of people have access denied everyday to a large variety of goods and services that are too expensive for their own budgets, although the markets could efficiently provide them at low prices. The individuals who afford such expensive goods are forced to pay for them a greater fraction of their income and so they are able to buy less of alternative goods. The downstream companies that rely on intermediate products supplied by a cartel face higher costs and are forced to charge higher prices to their customers as well, having less money available to distribute between their workers, administrators and shareholders. By reducing the level of output transacted, cartels are also a source of low employment levels and even low wages, particularly among highly skilled workers who have invested considerable time and money on formation and training. But worse of all, because collusion is not a zero sum game, the gains earned by cartelized industries are far exceeded by the damage imposed on the costumers, workers, administrators and shareholders of all honorable firms that are somehow affected by cartels, causing society to lose as a whole in a manner that can hardly by measured.

Unlike more traditional crimes as murder, robbery, bribery, extortion and fraud, which affect the integrity and wealth of individuals in a very perceivable way, the costs of collusion are dispersed over a large number of victims who rarely realize they have been injured at all, making collusion extremely hard to detect. Even so, every time a cartel fixes prices or restrains competition in the internal markets, a trace is left in the pattern of economic data that can be tracked by proper statistical tools. Therefore, in the same way murders and robberies are investigated with advanced techniques of forensic science to analyze DNA, fingerprints, footwear impressions and blood spatter, it is possible to develop analogous advanced econometric methods to screen the data and to constantly seek evidence of collusion.

While the empirical analysis of economic data may be extremely useful to uncover cartels that have successfully remained in secret so far, it is important to keep in mind that these methods cannot be used as hard evidence to prove guilt in the court of law. Nevertheless, we believe they can still be applied as, using the wording of Harrington (2005), a screening and verification device to identify the industries worthy of further investigation and prosecution. It should be the main purpose of these methods to improve an efficient allocation of antitrust authorities' resources towards the industries whose likelihood of collusion is higher and whose estimated costs on welfare are more severe, contributing to an increasing number of uncovered cartels. Moreover, in case hard evidence is collected and firms are actually condemned, the same empirical methods can many times be used to estimate the overcharge of the cartel, in order to determine the adequate fee or sentence.

There are already some empirical models of collusion detection in the literature, which we divide in three main groups: the models based on statistical features of data, the ones based on structural breaks and those based on price-cost margins. The models in the first group attempt to identify features of data that are consistent with either competition or collusion, by testing, for example, if price levels are correlated after controlling for demand or cost factors. In this respect, Bajari and Ye (2003) test for the presence of collusion in procurement auctions conducted by construction firms of the seal coating industry in Midwest, between 1994 and 1998. Their methodology involves testing whether firms' bids are independent and exchangeable (that is, if a permutation of the costs of firms leads to a permutation of their bids), two properties that should be verified in a competitive bidding scenario. Similar tests were conducted earlier by Porter and Zona (1993) and Baldwin, Marshall and Richard (1997), who respectively studied bidder collusion by state highways construction firms in Long Island in the 80s and collusion between purchasers of timber of the Forest Service in Pacific Northwest, in the 70s. In our view, the main limitation of these models is that they are narrowed to detect collusion in auctions and they cannot be extended easily to the analysis of other industries, where properties like independence and exchangeability may fail to distinguish collusion from competition.

The second group of models searches for structural breaks in time series, which are usually observed at the moments cartels are either created or closed. As long as there is some *a priori* information that may suggest possible breakpoints in data resulting from the formation or closure of a cartel (for example, periods with a considerable number of entries, exits or mergers), structural changes can easily be checked using a Chow test (1960). If the econometrician does not have any clues about possible breakpoints in data, a structural change in the whole time series can still be sought by following, for instance, the test proposed by Quandt (1960). In any case, one must be very cautious with this approach, since the structural break may be triggered by other reasons than collusion. A much more refined technique based on structural breaks was followed by Porter (1983) to identify the periods of time at which the Joint Executive Committee was operational, a famous railroad cartel in the 1880s that faced alternative periods of cooperative and non-cooperative behavior. His method consists in the joint estimation of two simple linear equations, a homogeneous product market demand and a switching regression characterizing the supply relationship of the industry with two possible regimes: collusion and competition. Using a version of the EM algorithm for the estimation of switching regressions, Porter was able to identify the regime observed in each period. Since then some authors have extended Porter's model to other industries, as Almoguera, Douglas and Herrera (2007), who studied the Organization of Petroleum Exporting Countries (OPEC) between 1974 and 2004 and tested for switches between collusive and non-cooperative periods.

The third group of models measures price-cost margins or other performance indices to access the degree of market power in the industry and to predict collusion. The relation between market performance and market conduct goes back to Bain (1951), who is credited for the development of the *structure, conduct and performance paradigm* that has dominated empirical industrial economics in the second half of the twentieth century. However, attempting to compare price-cost margins across markets or industries to detect possible cartels has considerable shortcomings. Foremost, there is rarely good cost data available in most economic databases and so these models are required to use indirect methods based on cost estimations (in this respect, see Bresnahan, 1989). But even more importantly, since price-cost margins depend on so many economic factors as product's characteristics, degree of differentiation, market size, patents, barriers to entry and regulation, high margins are not exclusively observed in cartels, as there are many profitable industries without any evidence of collusion. Instead of looking to the whole size of margins, a more modern approach uses economic theory to decompose the observed price-cost margins in unilateral and coordinated effects, and then to test whether the later is statistically significant. A model of this kind usually involves the estimation of demand functions and the mathematical computation of several price competition equilibria, in order to determine the unilateral effect of the price-cost margin that results from product differentiation, the unilateral effect that comes from market structure and the coordinated effect that results from collusion. This approach was initially developed by Nevo (2001) to show the absence of coordinated strategies in the breakfast cereal industry in the United States and it was then followed by Slade (2004) to study the brewing industry in the United Kingdom. One must keep in mind though that the coordinated effect estimated highly depends on the hypothesis that the behavior of the firms is accurately described by the theoretical models used to compute the different equilibria.

Despite the wide range of empirical models to detect collusion in the economic literature, so far none has been systematically used by competition authorities, whose investigation continues to rely mainly on consumer complaints. In reality, most models are still too complex, hard to implement, require the collection of a lot of data and must be adjusted case-by-case, creating a great problem for competition authorities who lack the time and resources to investigate every industry in such detail. Furthermore, there is still little evidence that those models are able, indeed, to accurately distinguish competition from collusion, once they have not been rigorously tested in the two distinct scenarios. For that reason, it is our purpose not only to develop a parsimonious and computationally efficient algorithm that can be easily applied by antitrust authorities to detect collusion, but also to prove that our approach is robust and accurate.

The method proposed in this thesis belongs to the group of models based on structural breaks and is related to Porter (1983), involving the estimation of the supply side of the industry as a switching regression. There are two important features that distinguish our model from Porter's. Firstly, we do not estimate the demand equation, which is particularly useful when we do not know its functional form or we do not observe many demand side variables. Secondly, our switching regression is estimated with a modified expectation-maximization (EM) algorithm that only uses analytical estimators (like TSLS) at each iterative step. This makes our method fast, easy to compute and less sensitive to initial points.

Using simulated data, we show that our algorithm is able to accurately predict collusion and, under specified conditions, to obtain consistent and unbiased estimates for the parameters of the switching regression. We also show that the algorithm is able to correct any estimation bias that may result from the misidentification of the regimes and from the identification problem of the supply equation. Resorting to simulated data has the enormous advantage of knowing exactly when firms are colluding and having access to the real parameters of the population, which enable us to determine the success rate of the method and to evaluate how close the estimates obtained are to the true underlying values. A similar procedure was followed by Paha (2011), who simulated an industry in order to evaluate different empirical methods to determine the overcharge of cartels. In addition, generating our own data allows us to perform sensitivity analysis to study how results are affected by changes in the parameters or even to create multiple samples with very large number of observations, in order to check statistical properties such as consistency and unbiasedness.

In the next section we distinguish different types of switching regressions and explain how they can be applied as a method of collusion detection. It is also shown how the switching regression that suits best our interests can be estimated using an expectation-maximization algorithm. In Section 4.3 we identify the estimation bias that results from the misidentification of the regimes and we explain how it can be avoided. In Section 4.4 we discuss the identification problem of the supply equation due to the endogeneity of the quantity transacted. In Section 4.5 we present a new EM algorithm which solves the identification problem and, under most conditions, is able to estimate consistently the parameters of the switching regression. Section 4.6 discusses a statistical test for structural breaks. Lastly, Section 4.7 concludes.

4.2. Switching regressions and the EM algorithm

Switching regression models have been increasingly applied to the analysis of various economic problems where the variable of interest is a function of structural parameters that vary across two or more regimes. When studying the cooperative behavior between firms, the variable of interest is usually the price and the two regimes are competition and collusion. In a simple industry where firms are price setters and the product is homogeneous, the supply side of the market would be described typically by a switching regression of the type

$$P_t = \begin{cases} MC_t + \beta_c Q_t + u_t^c, & if\ regime_t = collusion \\ MC_t + \beta_n Q_t + u_t^n, & if\ regime_t = competition\ , \end{cases} \tag{4.2.1}$$

where P_t is the price, MC_t is the marginal cost, Q_t is the quantity produced and u_t^c and u_t^n are normal unobserved errors. If we were able to observe the regime in action at time t, this model could be easily solved by separately estimating the two previous regression by ordinary least squares. Alternatively we could create a dummy variable S_t that takes the value 1 in collusive periods and 0 otherwise and estimate by *OLS* the regression:

$$P_t = MC_t + \beta_c Q_t S_t + \beta_n Q_t (1 - S_t) + u_t. \tag{4.2.2}$$

Notwithstanding, the regime operating at time t is not observed and we face the more complex challenge of estimating not only the structural parameters β_c and β_n, but also the state variable S_t. One possible approach to overcome such problem is frequently attributed to Goldfeld and Quandt (1972) and it involves seeking for an additional set of exogenous variables $\boldsymbol{z_t}$ that can somehow predict the regime observed each period. Following this methodology, the previous switching regression model would be rewritten as

$$P_t = \begin{cases} MC_t + \beta_c Q_t + u_t^c, & if\ \boldsymbol{z_t}'\boldsymbol{\pi} > 0 \\ MC_t + \beta_n Q_t + u_t^n, & if\ \boldsymbol{z_t}'\boldsymbol{\pi} \leq 0 \end{cases} \tag{4.2.3}$$

and the unknown parameters could now be estimated by maximum likelihood or by a Tobit model. Yet we do not always have many variables available that can accurately explain the cooperative behavior of firms along time. In fact, it is precisely our problem to detect periods of collusion without any other information but the data available on the supply side of the industry. And so, while Goldfeld and Quandt's method could be useful to estimate the parameters of the supply function if we already had some way to distinguish collusion from competition, it is our main goal to create such detecting mechanism.

Another approach that suits better our purposes was proposed by Quandt (1972), who presented a new switching regression model where the observations are generated by each regime with a constant but unknown probability:

$$P_t = \begin{cases} MC_t + \beta_c Q_t + u_t^c, & with\ probability\ \lambda \\ MC_t + \beta_n Q_t + u_t^n, & with\ probability\ 1-\lambda. \end{cases} \tag{4.2.4}$$

Here λ is the probability of collusion and $1-\lambda$ the probability of competition. This problem is also identified sometimes in the literature as a mixture of normal distributions, once the explained variable price is generated by two normal distributions with different means and variances. Kiefer (1978) proved that there is an unique consistent and asymptotically efficient estimator for the coefficients of a switching regression of this kind, which corresponds to a local maximum of the likelihood function

$$L = \prod_{t=1}^{T} \lambda\, pdf(P_t|S_t = 1) + (1-\lambda)\, pdf(P_t|S_t = 0), \tag{4.2.5}$$

where pdf is the probability density function of the normal variable P_t conditional on that a specific regime was observed. Nevertheless, because the maximizing conditions of the likelihood function above are non-linear and have several roots, it is not easy to detect which of them corresponds to our consistent estimator. One common solution is to use an expectation-maximization (EM) algorithm, an iterative method that is usually capable of successfully converging to the consistent root and which we will now briefly explain.

Consider the switching regression in (4.2.4) rewritten as

$$P_t = MC_t + \beta_c Q_t S_t + \beta_n Q_t (1 - S_t) + u_t, \qquad S_t = \begin{cases} 1\ with\ prob.\ \lambda \\ 0\ with\ prob.\ 1-\lambda \end{cases} \tag{4.2.6}$$

and define W_t as the probability of $S_t = 1$ conditional on the observation of P_t,

$$W_t = P\ (S_t = 1|P_t). \tag{4.2.7}$$

Given an initial guess for the unobserved variable W_t, it is relatively easy to obtain initial estimates for the parameters β_c, β_n and σ by maximizing, for instance, the likelihood function

$$L = \prod_{t=1}^{T} pdf(P_t|W_t), \qquad P_t \sim N(MC_t + \beta_c Q_t W_t + \beta_n Q_t(1 - W_t), \sigma). \quad (4.2.8)$$

Alternatively, one can follow Kiefer (1980) and simply estimate by least squares the regression

$$P_t = MC_t + \beta_c Q_t W_t + \beta_n Q_t (1 - W_t) + u_t. \quad (4.2.9)$$

This corresponds to the maximization step of the EM algorithm. The initial guesses for W_t can be used further to compute the unconditional probability of collusion λ as the mean of all conditional probabilities:

$$\lambda = \sum_{t=1}^{T} W_t. \quad (4.2.10)$$

Next, at the expectation step of the algorithm we must revise our expectations of the conditional probability of collusion W_t, taking into account the estimates obtained in the previous step. Using Bayes rule,

$$W_t = P(S_t = 1|P_t) = \frac{P(S_t = 1)\, P(P_t|S_t = 1)}{P(S_t = 1)\, P(P_t|S_t = 1) + P(S_t = 0)\, P(P_t|S_t = 0)} \leftrightarrow$$

$$W_t = \frac{\lambda\, pdf(P_t|S_t = 1)}{\lambda\, pdf(P_t|S_t = 1) + (1 - \lambda)\, pdf(P_t|S_t = 0)}. \quad (4.2.11)$$

Given the new values for W_t, new estimates for the parameters β_c, β_n, σ and λ can be obtained. The maximization and expectation steps are then iteratively repeated until convergence is reached (usually the algorithm stops when the coefficient of correlation between two successive estimates of W_t series is near one).

Kiefer (1980) proves that the estimates obtained by the EM algorithm correspond, in fact, to a local maximum of the likelihood function in (4.2.5). In the next section we will discuss the conditions under which the solution obtained is the consistent estimator and, otherwise, how we can deal with the estimation bias.

4.3. Dealing with the estimation bias

In order to evaluate the capability of the EM algorithm to produce the consistent estimators of the switching regression in (4.2.6), we conduct several simulation experiments with sets of data randomly generated from a very simple model of the industry. The variables marginal cost MC_t and quantity transacted Q_t are assumed to be exogenous and they are randomly drawn from two normal distributions. The state variable S_t is a random variable drawn from a *Bernoulli* distribution that takes value 1 (collusion) with success probability λ. The observations for the dependent variable price (P_t) are generated by equation (4.2.6).

We simulate seven different types of populations or industries, whose true underlying parameters are listed in Table 1. For each population we extract a sample of 10 000 observations and next we attempt to estimate consistently the parameters using a version for the EM algorithm we have written for Matlab and which is available in Appendix I. To guarantee that our results are not occasional, we replicate the simulation and estimation procedures 50 times for each population. Table 2 displays the estimates of the parameters obtained in average for the 50 replications.

Table 1 – Parameters of the population in simulation 1

Population	$\boldsymbol{\beta_c}$	$\boldsymbol{\beta_n}$	$\boldsymbol{\sigma}$	$\boldsymbol{\lambda}$
1	0.5	0.25	1	0.2
2	0.5	0.25	2	0.2
3	0.5	0.25	3	0.2
4	0.5	0.25	4	0.2
5	0.5	0.25	5	0.2
6	0.5	0.25	5	0.8
7	0.65	0.10	5	0.2

The first five populations in study have the same underlying parameters except for an increasing standard deviation of the error term, which clearly has a very important role in the consistency of the results. In fact, while in the first two populations the estimates obtained are very accurate and nearly equal to the true parameters, when the volatility of the error is raised we start observing a growing estimation bias, which is particularly evident in population 5. We observe that the parameters σ and λ are both under estimated, while the difference between the parameters β_c and β_n is over estimated.

Table 2 – Average estimates of the OLS EM algorithm

Population	$\widehat{\boldsymbol{\beta}}_{\boldsymbol{c}}$	$\widehat{\boldsymbol{\beta}}_{\boldsymbol{n}}$	$\widehat{\boldsymbol{\sigma}}$	$\widehat{\boldsymbol{\lambda}}$	$\boldsymbol{R^2}$
1	0.5001	0.2500	0.9988	0.1992	0.9650
2	0.5015	0.2506	1.9755	0.1998	0.8763
3	0.5102	0.2460	2.7407	0.1995	0.7948
4	0.5371	0.2441	3.3953	0.1915	0.7346
5	0.5798	0.2408	4.1619	0.1651	0.6701
6	0.5042	0.1657	4.1542	0.8330	0.6831
7	0.6528	0.0965	4.8657	0.1998	0.8406

To understand why estimates for populations 3 to 5 turned out to be biased, we illustrate graphically the sources of the estimation bias in a very simple example. For that purpose, consider that the average price observed at the industry depends only on the regime of interaction between firms, collusion or competition. That is, the price in period t is a switching regression of a constant term:

$$P_t = \alpha_c S_t + \alpha_n (1 - S_t) + u_t, \qquad S_t = \begin{cases} 1 \; with \; prob. \; \lambda \\ 0 \; with \; prob. \; 1 - \lambda \end{cases}. \tag{4.3.1}$$

Consider further that we only have available the four observations represented in Figure 1, of which the first two correspond to a collusive regime and the last two to a competitive regime. In this example, any reasonable estimation technique would be able to correctly distinguish the two regimes and to accurately estimate each constant term as the average value of the two corresponding observations, leading to consistent results.

However, suppose now that the error term of equation (4.3.1) has a much higher standard deviation and that we observe instead the four occurrences in Figure 2, of which the first two correspond again to collusion and the last two to competition. Although according to Kiefer (1980) the likelihood function of the switching regression still has a local optimal at the point where the estimates are exactly equal to α_c and α_n, there is now a much more obvious global optimal to which any iterative algorithm will converge. Given that only the four dots in Figure 2 can be observed, our best guess is that the two higher occurrences of P_t (at periods 2 and 4) are observed under collusion and the two lower occurrences (at periods 1 and 3) are observed under competition. And so the algorithm will estimate the two constant terms as the mean of those two pairs of observations, which are given by the dashed lines in graphic 2, leading therefore to estimation bias.

Figure 1 – Identification of the regimes

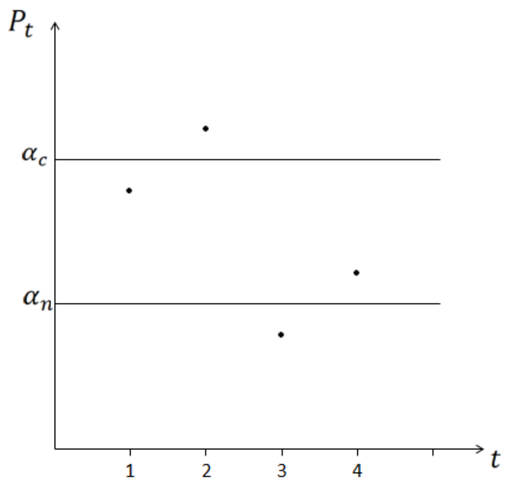
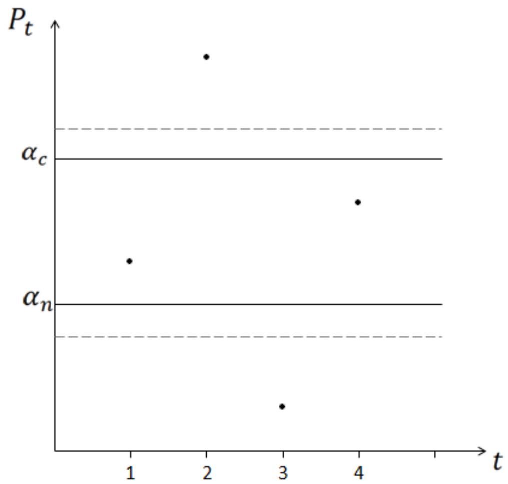


Figure 2 – Misidentification of the regimes



We conclude from the example above that in switching regression models where the error term is too volatile, the iterative algorithm fails to identify the regime in some periods. Intuitively, when under collusion the price observed is very high, we are able to correctly identify the operative regime, but when the price is particularly low we misidentify the regime as competition. As a result, average collusive prices appear to be higher than what they are in reality and the parameters associated with collusion are overestimated. The opposite occurs at the competitive regime, whose average prices and parameters tend to be underestimated.

In addition, because the lower prices under collusion and the higher prices under competition are classified in the opposite regime, some of the residuals become considerably smaller than the true errors of the population and thus the standard deviation σ is also underestimated, as it is observed in Table 2.

Finally, because the standard deviation σ affects the probability density function of P_t, it follows directly from equations (4.2.10) and (4.2.11) that the conditional probabilities of collusion W_t and the unconditional probability λ are inconsistently estimated as well. The direction of the estimation bias depends on the true value of λ. As we can see in

Table 2, λ is underestimated in populations 3 to 5, where the competitive regime is operational in most observations ($\lambda = 0.2$). But in population 6, where the true value of λ is 0.8 and collusion dominates the sample, λ is overestimated.

In conclusion, despite the existence of a consistent root that maximizes locally the likelihood function in (4.2.5), when the switching regression model is not robust and the error has too much volatility, the consistent root is not tractable, once it is not possible to distinguish properly the two regimes. Any other factor that affects the overall quality of the model leads to the same problem. For instance, because population 7 has a higher α_c and a lower α_n than population 5 (while all the remaining parameters are the same), the explained part of the model is greater and the estimates obtained are consistent. Everything else equal, the greater the R^2 is, the greater is the ability of the algorithm to converge to the consistent root of the switching regression model. In our simulation experience, for an R^2 greater than 0.88 the estimates appear to be consistent and for an R^2 around 0.79 the estimation bias is not significant.

Since it is not always possible to estimate a model with an R^2 above 0.8, it would be useful to somehow improve the ability of the EM algorithm to detect the consistent root when the model is not so strong. For that reason, we propose a small correction in the iterative process to obtain better results. Recall that in each maximization step of the version of the EM algorithm in Kiefer (1980), the beta coefficients are estimated given the conditional probabilities W_t obtained in the previous expectation step. That is, the maximization step computes the least squares estimates of the regression:

$$P_t = MC_t + \beta_c Q_t W_t + \beta_n Q_t (1 - W_t) + u_t. \tag{4.3.2}$$

In our alternative version of the algorithm (see Appendix J), we propose to estimate the unknown coefficients taking into account our better prediction of the regime operating at time t or, in other words, to estimate the following regression by least squares:

$$P_t = MC_t + \beta_c Q_t \hat{S}_t + \beta_n Q_t \left(1 - \hat{S}_t\right) + u_t, \qquad where\ \hat{S}_t = \begin{cases} 1 \ \ if\ W_t \geq 0.5 \\ 0 \ \ if\ W_t < 0.5 \end{cases}. \tag{4.3.3}$$

By replacing the conditional probabilities W_t, which range between 0 and 1, by the predicted regimes $\hat{S}_t$, which can only assume the extreme values 0 or 1, it is our intention to avoid underestimating the standard deviation of the error. In fact, when the probabilities W_t are introduced in equation (4.3.2), the fitted prices depend on a linear combination of the structural parameters of the two regimes and so the residuals become much smaller than the true errors, especially in observations whose prices are somewhere between the collusive and the competitive expected level. But when we use a variable $\hat{S}_t$ in equation (4.3.3) that can only assume the value 0 or 1, the fitted prices are located at one of the two extremes and the residuals are closer to the true errors.

In order to address the quality of the new algorithm, we estimated again the coefficients of the seven populations with the same data samples used to compute the values in Table 2. The new estimation results are displayed in Table 3.

Table 3 – Average estimates of the modified OLS EM algorithm

Population	$\widehat{\boldsymbol{\beta}}_c$	$\widehat{\boldsymbol{\beta}}_n$	$\widehat{\boldsymbol{\sigma}}$	$\widehat{\boldsymbol{\lambda}}$	$\boldsymbol{R^2}$
1	0.5001	0.2500	0.9988	0.1992	0.9650
2	0.5007	0.2506	1.9975	0.1998	0.8735
3	0.4998	0.2473	2.9369	0.2000	0.7643
4	0.5086	0.2453	3.7928	0.1943	0.6689
5	0.5257	0.2406	4.6735	0.1756	0.5841
6	0.5051	0.2205	4.6729	0.8224	0.5991
7	0.6477	0.0970	4.9867	0.1998	0.8325

It is remarkable that the estimates obtained by the new version of the EM algorithm are considerably closer to the true underlying parameters, with the exception of population 1, whose estimates were already consistent. As we can observe in Table 3, the new algorithm converges now to the consistent estimator of the switching regression for an R^2 higher than 0.76 and the estimation bias is still very small for an R^2 around 0.67.

Despite the considerable improvement in our estimation results, if it is our desire to run switching regressions with actual data and to detect real cartels, further modifications must be implemented in the algorithm in order to solve the identification problem. We address this issue in the next section.

4.4. The identification problem

So far, for exposition purposes, we have discussed the estimation of a switching supply function in a very simplified scenario, where the variable price is endogenously set by firms and the quantity transacted is exogenous. Unfortunately, the later assumption does not usually hold in reality, since it is well established by economic theory that prices and quantities are both endogenously determined by a simultaneous system of equations composed by the market demand and supply functions. For that reason, we will now focus our analysis on the estimation of a more realistic model of the industry, where prices and quantities observed at time t are the solution of the following system:

$$\begin{cases} Q_t = \beta_0 + \beta_1 Y_t + \beta_2 P_t + v_t \\ P_t = MC_t + \beta_c Q_t S_t + \beta_n Q_t (1 - S_t) + u_t \end{cases} . \tag{4.4.1}$$

The first equation is the market demand, in which Y_t is the income. v_t and u_t are the error terms of the demand and supply functions, with zero mean and constant standard deviation. All the other variables have the same meaning as in Section 4.2.

When prices and quantities are the solution of a system of this type, the econometrician typically observes a map of dispersed points as in Figure 3 and is not able to identify the demand and supply relationships by traditional estimation methods. To solve the identification problem, at least one exogenous variable must be observed in each equation of the system that does not appear in the other equation, like the income and the marginal cost in our example. This way, the econometrician can make sense of each price-quantity combination observed as a particular equilibrium that results from the dislocation of the demand and supply curves (Figure 4).

Figure 3 – Identification problem

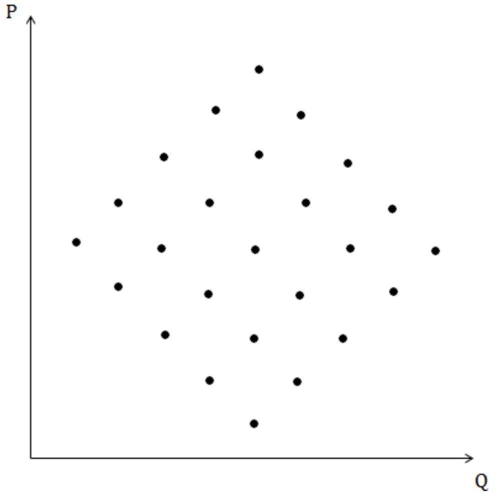
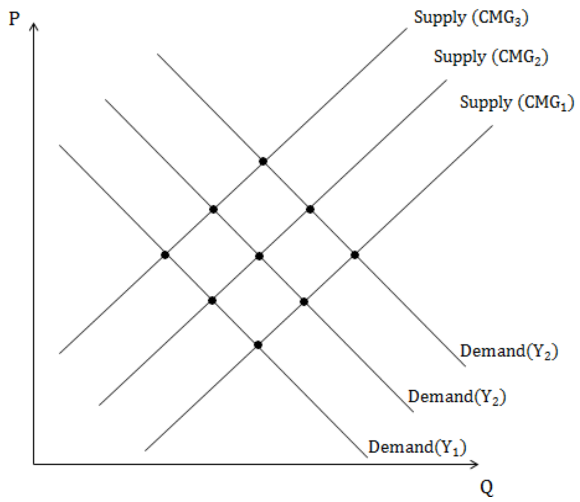


Figure 4 – Solving the identification problem



As long as the state variable S_t is observed data, there are several methods available to estimate some or all parameters of the system in (4.4.1). The most common procedure is either to estimate separately one of the equations using instrumental variables (generally by two-stages least squares – 2SLS) or to simultaneously estimate the whole system by three-stage least squares (3SLS). Alternatively one can use the limited information maximum likelihood (LIML) or the full information maximum likelihood (FIML) methods to obtain the analogous maximum likelihood estimators of the parameters in one or both equations.

On the other hand, when S_t is unknown as is our case, one of the methods above must be somehow combined with the switching regression techniques exposed in the earlier sections, turning the analysis much more complex. Such procedure was already conducted by Porter (1983), who successfully introduced the full information maximum likelihood method in the iterative steps of the EM algorithm, in order to identify the switching periods between collusion and competition of the Joint Executive Committee railroad cartel. However, we believe the estimation by full information maximum likelihood comes with several problems difficult to overcome. Firstly, the FIML estimation is computationally heavy, time consuming and extremely sensitive to initial points, making it hard to converge to the solution when the sample is large, the equations have atypical functional forms and if we do not have a good guess of the initial values. When applied to switching regressions it is even harder to converge, given that the FIML must be repeatedly run at each of the iterative steps of the EM algorithm. Secondly, the good properties of the FIML, like consistency and asymptotic efficiency, depend on the assumption that the distribution of the errors is well specified. And thirdly the FIML require us to estimate the whole system even if we do not have any clue about the functional form of the market demand or even if we have no interest in estimating it.

Therefore, while the estimation of a switching regression using FIML seems achievable for a professional researcher focusing entirely on the analysis of a single industry, it does not seem so attractive for competition authorities who have limited time and resources to investigate more than a few industries. For that reason we propose a new EM algorithm that solves instead the identification problem using two-stage least squares, a simple and parsimonious analytical method that is easy to compute, has good properties and does not rely on much information *a priori*. In the following section we will briefly discuss the main modifications that need to be introduced in the maximization and expectation steps, in order to estimate the switching supply function in the system in (4.4.1).

4.5. An efficient EM algorithm

Consider a dataset composed by two randomly generated time series, MC_t and Y_t, as well as two endogenous variables P_t and Q_t that solve the system in (4.4.1). Suppose further it is our purpose to estimate only the switching supply function:

$$P_t = MC_t + \beta_c Q_t S_t + \beta_n Q_t (1 - S_t) + u_t. \tag{4.5.1}$$

At the maximization step of the EM algorithm we must estimate the coefficients β_c, β_n and σ for some given expectations of the regime variable S_t. Notwithstanding, it results from the demand equation that the market price has a feedback effect on the quantity that consumers are willing to buy, leading to correlation between Q_t and u_t. As the OLS estimators are no longer consistent, we show how TSLS can be successfully employed to estimate the three parameters. Foremost, we split the system in (4.4.1) in two different subsystems, for $S_t = 1$ and for $S_t = 0$:

$$\begin{cases} Q_t = \beta_0 + \beta_1 Y_t + \beta_2 P_t + v_t \\ P_t = MC_t + \beta_c Q_t + u_t \end{cases}, \qquad if\ S_t = 1 \tag{4.5.2}$$

$$\begin{cases} Q_t = \beta_0 + \beta_1 Y_t + \beta_2 P_t + v_t \\ P_t = MC_t + \beta_n Q_t + u_t \end{cases}, \qquad if\ S_t = 0. \tag{4.5.3}$$

Solving each system with respect to P_t and Q_t gives the reduced form equations for the quantity transacted as a function of the exogenous variables:

$$Q_t = \alpha_c + \gamma_c MC_t + \delta_c Y_t + z_{ct}, \qquad if\ S_t = 1 \tag{4.5.4}$$

$$Q_t = \alpha_n + \gamma_n MC_t + \delta_n Y_t + z_{nt}, \qquad if\ S_t = 0. \tag{4.5.5}$$

Then, at the first stage of TSLS we estimate equations (4.5.4) and (4.5.5) by OLS and compute the fitted values of the quantity transacted conditional on collusion ($\hat{Q}_{ct}$) and conditional on competition ($\hat{Q}_{nt}$):

$$\hat{Q}_{ct} = \hat{\alpha}_c + \hat{\gamma}_c MC_t + \hat{\delta}_c Y_t \tag{4.5.6}$$

$$\hat{Q}_{nt} = \hat{\alpha}_n + \hat{\gamma}_n MC_t + \hat{\delta}_n Y_t. \tag{4.5.7}$$

At the second stage, we replace the quantities transacted by their fitted values in the supply functions and we simply estimate the new equations by OLS:

$$P_t = MC_t + \beta_c \hat{Q}_{ct} + w_{ct}, \qquad if\ S_t = 1 \tag{4.5.8}$$

$$P_t = MC_t + \beta_n\, \hat{Q}_{nt} + w_{nt}, \qquad if\ S_t = 0. \tag{4.5.9}$$

Alternatively, we may run the next equation by OLS:

$$P_t = MC_t + \beta_c \hat{Q}_{ct} S_t + \beta_n \hat{Q}_{nt}(1 - S_t) + w_t. \tag{4.5.10}$$

Remark that the predicted values of $\hat{Q}_{ct}$ and $\hat{Q}_{nt}$ are a function of exogenous variables and are not correlated with w_t. For that reason, equation (4.5.10) can be run by OLS to consistently estimate β_c, β_n and σ.

At the expectation step of the new EM algorithm we must revise our expectations of the regime observed at time t, given the estimates obtained for the coefficients. Once there are now two endogenous variables, if we were able to estimate both the supply and demand equations of the system in (4.4.1), our best prediction of the likelihood of observing collusion at time t would be the probability that S_t was equal to one, conditional on the observations of P_t and Q_t. Using Bayes rule:

$$W_t = P(S_t = 1 | P_t \cap Q_t) =$$

$$= \frac{P(S_t = 1)\, P(P_t \cap Q_t | S_t = 1)}{P(S_t = 1)\, P(P_t \cap Q_t | S_t = 1) + P(S_t = 0)\, P(P_t \cap Q_t | S_t = 0)} =$$

$$= \frac{\lambda\, P(P_t \cap Q_t | S_t = 1)}{\lambda\, P(P_t \cap Q_t | S_t = 1) + (1 - \lambda)\, P(P_t \cap Q_t | S_t = 0)}. \tag{4.5.11}$$

Howsoever, since we are assuming we do not have enough information to estimate the demand equation, it is not possible to compute $P(P_t \cap Q_t | S_t = 1)$ and $P(P_t \cap Q_t | S_t = 0)$, simply because we do not know the density function of the variable Q_t that results from the demand equation. Instead we must determine the likelihood of observing collusion at time t as the probability that S_t is equal to one, conditional only on the observation of P_t:

$$W_t = P(S_t = 1 | P_t) = \frac{\lambda\, P(P_t | S_t = 1)}{\lambda\, P(P_t | S_t = 1) + (1 - \lambda)\, P(P_t | S_t = 0)}. \tag{4.5.12}$$

As before, λ is calculated as the average of the conditional probabilities W_t obtained in the previous iteration of the algorithm. As regards to the probabilities of P_t conditional on a particular regime, $P(P_t | S_t = 1)$ and $P(P_t | S_t = 0)$, they can be derived from the probability density functions of P_t that result from the supply functions in (4.5.8) and (4.5.9):

$$P_t \sim N(\mu_{ct}, \sigma_{w_c}), \qquad \mu_{ct} = MC_t + \beta_c\, \hat{Q}_{ct}, \qquad if\ S_t = 1 \tag{4.5.13}$$

$$P_t \sim N(\mu_{nt}, \sigma_{w_n}), \qquad \mu_{nt} = MC_t + \beta_n\, \hat{Q}_{nt}, \qquad if\ S_t = 0. \tag{4.5.14}$$

It is very important to note that the expected price at time t conditional on a specific regime, μ_{ct} or μ_{nt}, depends on the fitted quantity transacted for that regime and not on the quantity that was actually observed. For instance, if firms compete in period t and we want to estimate the price that would be observed at that time if they had colluded, we must account that collusion affects the quantity transacted as well (by increasing the price, it reduces the total quantity consumers are willing to buy). And so, the expected price under collusion at time t can be obtained from the collusive supply function, evaluated at the quantity that is predicted under the same regime – equation (4.5.8). The same is true for the expected price under competition. This reasoning is a natural extension of the mechanism behind the TSLS estimation to the expectation step of the EM algorithm.

We have introduced the maximization and expectation steps previously described in a new version of the EM algorithm, which must now be tested using simulated data (the *Matlab* code is available in Appendix K). We consider six different types of populations, whose true underlying parameters are listed in Table 4 and whose prices and quantities are the solution of the system in (4.4.1). The six populations share the same beta and lambda coefficients, but have different standard deviations for the error terms.

Table 4 – True parameters of the populations in simulation 2

Pop.	$\boldsymbol{\beta_0}$	$\boldsymbol{\beta_1}$	$\boldsymbol{\beta_2}$	$\boldsymbol{\beta_c}$	$\boldsymbol{\beta_n}$	$\boldsymbol{\sigma_v}$	$\boldsymbol{\sigma_u}$	$\boldsymbol{\lambda}$
1	50	5	-2	0.5	0.25	1	1	0.2
2	50	5	-2	0.5	0.25	2	2	0.2
3	50	5	-2	0.5	0.25	3	3	0.2
4	50	5	-2	0.5	0.25	4	4	0.2
5	50	5	-2	0.5	0.25	1	4	0.2
6	50	5	-2	0.5	0.25	4	1	0.2

For each population we extract 50 random samples of 10 000 observations, which are used to compare the estimation results of the OLS EM algorithm with the estimation results of the new version of the EM algorithm based on TSLS. In Tables 5 and 6 we present the estimates obtained in average for the 50 replications using the two algorithms.

From the observation of the first four populations in Table 5 we verify that the OLS EM algorithm systematically underestimates the two coefficients of the supply function, β_c and β_n, as well as the probability of collusion λ. The estimation bias grows exponentially when the standard deviation of the errors increases. Fortunately, the results obtained for the same four populations in Table 6 show that the TSLS EM algorithm successfully removes the estimation bias originated by the identification problem. Indeed, for the two models with an R^2 above 0.87 the estimates are very accurate and for an R^2 above 0.74 the estimation bias is almost irrelevant. Notice further that the relatively small estimation bias observed in the third and forth lines of Table 6 do not result from the endogeneity of the quantity transacted, but from the inability of the algorithm to identify the correct regime in some periods. This can be concluded from the fact that the estimates of β_c and β_n diverge from each other, in opposition to Table 5 where the two coefficients are both underestimated.

Finally, it is possible to conclude from the results for population 5 and 6 that the error term of the supply function is the main source of the estimation bias caused by the identification problem, while the error of the demand function does not appear to be very relevant.

Table 5 – Average estimates of the modified OLS EM algorithm

Population	$\hat{\beta}_c$	$\hat{\beta}_n$	$\hat{\sigma}_u$	$\hat{\lambda}$	R^2
1	0.4941	0.2455	0.9958	0.1993	0.9665
2	0.4752	0.2297	1.9635	0.1908	0.8761
3	0.4464	0.2004	2.9077	0.1658	0.7457
4	0.4217	0.1625	3.9404	0.1188	0.5721
5	0.4206	0.1607	3.9515	0.1155	0.5620
6	0.4942	0.2456	0.9961	0.2000	0.9672

Table 6 – Average estimates of the TSLS EM algorithm

Population	$\widehat{\boldsymbol{\beta}}_c$	$\widehat{\boldsymbol{\beta}}_n$	$\widehat{\boldsymbol{\sigma}}_u$	$\widehat{\boldsymbol{\lambda}}$	$\boldsymbol{R}^2$
1	0.4999	0.2498	0.9986	0.2001	0.9663
2	0.4993	0.2489	1.9813	0.1997	0.8738
3	0.5004	0.2465	2.9213	0.1960	0.7434
4	0.5085	0.2424	3.8559	0.1771	0.5903
5	0.5087	0.2413	3.7393	0.1870	0.6079
6	0.4987	0.2496	1.0624	0.1993	0.9627

4.6. Testing for structural breaks

We have developed a fast and parsimonious algorithm to consistently estimate the supply side of an industry as a regression that switches between two regimes, competition and collusion. All the previous analysis was developed under the assumption that structural breaks do exist, that is, it was assumed that each regime is observed at least in some periods. A relevant question that must now be addressed is how the algorithm behaves when no structural break occurs (λ is equal to zero or one).

Recall the switching regression in equation (4.5.1) and rewrite it in matrix notation:

$$\boldsymbol{P} = \boldsymbol{X\beta} + \boldsymbol{U},$$

$$where\ \boldsymbol{X} = \begin{bmatrix} MC_1 & Q_1S_1 & Q_1(1-S_1) \\ \dots & \dots & \dots \\ MC_T & Q_TS_T & Q_T(1-S_T) \end{bmatrix} \quad and \quad \boldsymbol{\beta} = \begin{bmatrix} 1 \\ \beta_c \\ \beta_n \end{bmatrix}. \tag{4.6.1}$$

When firms either compete or collude along the whole time series, the variable S_t or $(1 - S_t)$ is a row of zeros and the matrix of regressors $\boldsymbol{X}$ is singular. Therefore, if the EM algorithm is able to converge to the true values of the state variable, it will attempt at some point to invert a singular matrix during the maximization step, generating thus an error.

Interestingly, the algorithm often converges instead to a meaningless random solution, estimating a value for lambda between zero and one and identifying both regimes in the dataset. Indeed, despite the absence of structural breaks, the EM algorithm allows the regime to switch along time in order to improve the fitting of the data, attributing the observations with high errors to collusion and the observations with low errors to competition. Unfortunately this means we cannot actually rely on the estimation output of the switching regression, unless we know for sure that the data was generated by a mixture of the two regimes. To overcome this problem we must implement some statistical test to check whether there is evidence for structural breaks.

We may be initially tempted to use a Wald test or a likelihood ratio test to verify whether the fitted state variable $\widehat{S_t}$ is statistically significant, case in which we conclude the time series is a mixture of two distinct regimes. However, the test statistics of the switching regression model do not have the traditional distributions under the null and cannot be used to conduct such analysis. For that reason, several authors have proposed a modified likelihood ratio test to test the hypothesis of a homogeneous model against a mixture of two or more regimes[7], as Chen and Kalbfleisch (2004) and Zhu and Zhang (2003). Still, these methods are quite hard to implement, as they involve finding lower and upper bounds and the asymptotic behavior of the distribution of the test statistic.

Here we propose a much more simple and convenient approach. Instead of studying the extremely complex distribution of the switching regression's estimator and test statistics, we focus our analysis on the simple TSLS estimation output. Consider again a dataset generated by system (4.4.1) and suppose we estimate the following regression by TSLS, using Y_t as an instrumental variable:

$$P_t = MC_t + \beta Q_t + u_t. \tag{4.6.2}$$

On the one hand, if the regime remains unchanged along time, the TSLS estimator is consistent and it follows directly that the residuals have a normal distribution, as observed in Figure 5.

Figure 5 – Histogram of the residuals of a homogeneous model.

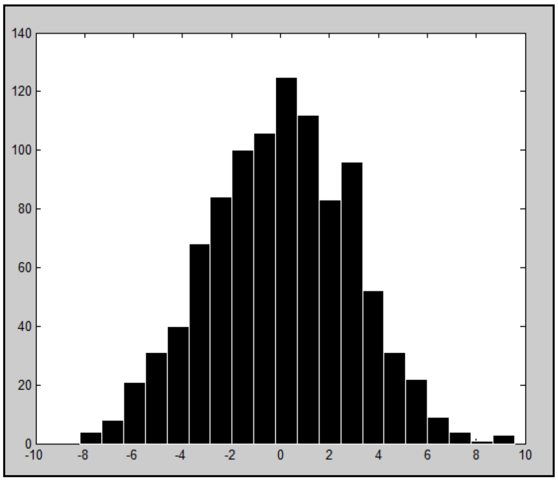
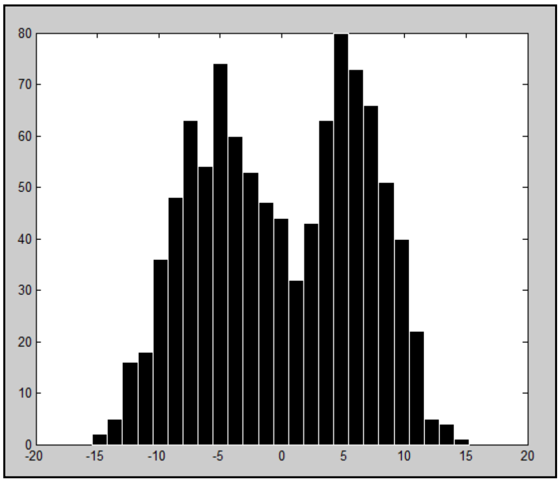


Figure 6 – Histogram of the residuals of a mixture model.



[7] For hypotheses tests of two against more regimes see Dannemann and Holzmann (2010).

On the other hand, if the time series has structural breaks separating the two regimes, the TSLS estimator is no longer consistent and the residuals are given by:

$$e_t = \begin{cases} (\beta_n - \hat{\beta}_{TSLS})Q_t + u_t, & if\ S_t = 0 \\ (\beta_c - \hat{\beta}_{TSLS})Q_t + u_t, & if\ S_t = 1 \end{cases}, \tag{4.6.3}$$

where $\hat{\beta}_{TSLS}$ is the TSLS estimator of regression (4.6.2). In other words, in the presence of structural breaks, the residuals are generated by a mixture of two normal distributions, as observed in Figure 6.

The clear distinction between the two distributions in the homogeneous and mixture models can be used to obtain evidence for structural breaks. For instance, we can test for normality of the residuals following Jarque and Bera (1987), whose null hypothesis is that the residuals follow a normal distribution with unknown mean and variance. If the null is rejected, it raises suspicions that there are structural breaks and we can then run the EM algorithm to verify whether the industry is well explained by a switching regression model.

We conduct several simulation experiments to evaluate the Jarque-Bera test as a tool to detect the presence of structural breaks in the time series. As before, we simulate an industry with random levels of income and marginal costs and we generate the prices and quantities transacted as a solution of the system (4.4.1). The true values of the parameters are as follows:

Table 7 – Parameters of the population in simulation 3

$\boldsymbol{\beta_0}$	$\boldsymbol{\beta_1}$	$\boldsymbol{\beta_2}$	$\boldsymbol{\beta_c}$	$\boldsymbol{\beta_n}$	$\boldsymbol{\sigma_v}$
50	5	-2	0.5	0.25	1

For each simulation experiment we collect a random sample of observations from the population, estimate equation (4.6.2) by TSLS to obtain the residuals and run the Jarque-Bera test for a 5% significance level. Then we replicate the previous steps several times and calculate the type I and type II errors, whose results are displayed in Table 8.

To calculate the type I error we set λ equal to 0, so that firms always compete and there are no structural breaks in the data. Next we compute the fraction of replications where the Jarque-Bera test rejects the null hypothesis of normality, suggesting structural breaks. To calculate the type II error we set λ equal to 0.2 to obtain a sample with a mixture of collusive and competitive periods. Then we compute the fraction of replications where the Jarque-Bera test fails to reject the null hypothesis, suggesting a homogeneous model.

Table 8 – Jarque-Bera test

Experiment	**Observations**	**Replications**	$\boldsymbol{\sigma_u}$	**Type I Error** $\lambda = 0$	**Type II Error** $\lambda = 0.2$
1	10 000	100	2	0.05	0
2	10 000	100	3	0.04	0
3	10 000	100	4	0.09	0
5	1000	500	2	0.06	0
6	1000	500	3	0.032	0
7	1000	500	4	0.044	0
9	200	500	2	0.05	0
10	200	500	3	0.05	0
11	200	500	4	0.048	0.236
12	100	500	2	0.054	0
13	100	500	3	0.05	0.124
14	100	500	4	0.058	0.666

From the observation of Table 8 we conclude that, for all experiments, the type I error is very close to the 5% significance level, meaning that the test is well elaborated and can be used, indeed, to check for structural breaks. In addition, from the analysis of the type II error, we conclude that the power of the test is enormous (equal to one) when the number of observations is very large. Howsoever, when we collect a smaller sample of 100 or 200 observations, the test may lose some power if the error term becomes too volatile. For instance, in experiment 13, for a sample of 100 observation and a standard deviation of the error equal to 3, the Jarque-Bera test fails to reject the null hypothesis of normality in 12,4% of the cases where the null is false. In order words, the power of the test is 0.876. The remaining experiments where the error type II is positive (experiments 11 and 14) are not so relevant because, as we saw in the previous sections, for an error term so erratic it is not possible to accurately distinguish the two regimes and the EM algorithm malfunctions anyway.

The ability of the Jarque-Bera test to detect structural breaks depends, of course, on the assumption that the error terms of the homogeneous and mixture models follow in fact a normal distribution (actually almost all statistical inference depend on that assumption). While this should not represent a major problem for large samples in which the central limit theorem can be applied, for small samples it may be useful to run alternative tests to check whether the results remain valid using different distributions, as the logistic.

In conclusion, the Jarque-Bera test appears to be an easy, fast and functional method to test for structural breaks. When it rejects the hypothesis of a homogeneous model, it can be complemented with the EM algorithm to verify if the industry can be accurately described by a switching regression of collusive and competitive regimes.

4.7. Conclusions

The pattern of economic data is the result of multiple complex interactions in the economy, some of which correspond to socially valuable transactions and others to criminal activities with high social costs. Modern econometric tools can be used to dissect the available data and detect some of those criminal behaviors that would otherwise remain unknown. In Chapter 4 we are concerned, in particular, with the empirical detection of collusion through switching regression models.

Although switching regressions have been broadly applied in various fields of social and natural sciences, the difficult estimation techniques behind them remain a black box for many researchers. We attempt here to get inside the black box in a controlled simulation environment in order to identify some common estimation problems, provide feasible solutions and present a better intuitive understanding of the results obtained.

Firstly, we show that the traditional expectation-maximization algorithm often fails to converge to the consistent root of the switching regression, due to the misidentification of the regime operating in some periods. To correct the estimation bias we propose an adjustment in the algorithm, although ultimately the switching regression model should have a sufficiently good explanatory power and the error should be relatively stable so that the results obtained are consistent.

Secondly, we address the problem of identifying the supply side of the industry when the data observed is the result of a system of supply and demand equations. We solve the identification problem by extending the rationality of the TSLS estimator to the maximization and expectation steps of the EM algorithm.

Thirdly, we show that estimating a switching regression alone does not provide absolute evidence that the time series is composed by a mixture of collusive and competitive regimes. To test whether there are, indeed, structural breaks in data, we propose the implementation of the Jarque-Bera normality test for the residuals. If the null hypothesis is rejected, we conclude for the presence of structural breaks and we can then consistently estimate the supply curve as a switching regression.

Most importantly, we believe the empirical methods presented in this chapter are easy to implement, computationally efficient and do not rely on much information *a priori*.

And so, they can be actually applied by competition authorities who face time and resources constrains. There is, of course, room for improving the empirical analysis of collusion. It would be useful if future research taught us, for example, how to deal with industries with differentiated products, firms with multiple products and more complex supply functions that account for other effects of collusion. Perhaps the continuous development of advanced econometric tools to detect collusion will turn competition authorities, over time, into real “collusive scene investigators”.

Appendices

Appendix A

Theorem: under convex cost functions and concave and additive demand functions, the price of any firm at labor market B is increasing with the price of any firm at labor market A.

Proof: the best reply function in equation (A.1) expresses the optimal price of any firm at labor market B as a function of the quantity produced, which in turn is a function of the prices of the whole industry:

$$P_i^B = \frac{\partial Total\ Cost_i^B}{\partial Q_i^B} - \frac{Q_i^B}{\partial Q_i^B / \partial P_i^B}. \tag{A.1}$$

Define BRF_i as the following function:

$$BRF_i = \quad P_i^B - \frac{\partial Total\ Cost_i^B}{\partial Q_i^B} + \frac{Q_i^B}{\partial Q_i^B / \partial P_i^B} = 0.$$

Using the implicit function theorem:

$$\frac{dP_i^B}{dP_j^A} = -\frac{\dfrac{\partial BRF_i}{\partial P_j^A}}{\dfrac{\partial BRF_i}{\partial P_i^B}} = -\frac{-\dfrac{\partial^2 Total\ Cost_i^B}{\partial {Q_i^B}^2}\dfrac{\partial Q_i^B}{\partial P_j^A} + \dfrac{\dfrac{\partial Q_i^B}{\partial P_j^A}\dfrac{\partial Q_i^B}{\partial P_i^B} - Q_i^B \dfrac{\partial^2 Q_i^B}{\partial P_i^B \partial P_j^A}}{(\partial Q_i^B / \partial P_i^B)^2}}{1 - \dfrac{\partial^2 Total\ Cost_i^B}{\partial {Q_i^B}^2}\dfrac{\partial Q_i^B}{\partial P_i^B} + \dfrac{\dfrac{\partial Q_i^B}{\partial P_i^B}\dfrac{\partial Q_i^B}{\partial P_i^B} - Q_i^B \dfrac{\partial^2 Q_i^B}{\partial {P_i^B}^2}}{(\partial Q_i^B / \partial P_i^B)^2}}.$$

Because the demand function is additive, $\frac{\partial^2 Q_i^B}{\partial P_i^B \partial P_j^A}$ is null and the convexity of the cost function and the concavity of the demand function imply that

$$\frac{dP_i^B}{dP_j^A} = \frac{\dfrac{\partial^2 Total\ Cost_i^B}{\partial {Q_i^B}^2}\dfrac{\partial Q_i^B}{\partial P_j^A} - \dfrac{\dfrac{\partial Q_i^B}{\partial P_j^A}\dfrac{\partial Q_i^B}{\partial P_i^B}}{(\partial Q_i^B / \partial P_i^B)^2}}{2 - \dfrac{\partial^2 Total\ Cost_i^B}{\partial {Q_i^B}^2}\dfrac{\partial Q_i^B}{\partial P_i^B} - \dfrac{Q_i^B \dfrac{\partial^2 Q_i^B}{\partial {P_i^B}^2}}{(\partial Q_i^B / \partial P_i^B)^2}} > 0.$$

and the assertion is complete.

From equation (A.1) it can be also concluded that the augment of P_j^A causes Q_i^B to rise as well. Once a higher P_i^B increases the left-hand side and decreases the right-hand side of equation (A.1), the equality must be restored at a higher Q_i^B.

Appendix B

Theorem: under trigger strategies, as long as every firm is strictly better off when colluding, there are discount rates lower than one for which collusion can be sustained in equilibrium at every stage of an infinite super game.

Proof: According to trigger strategies, each firm sets the collusive price or wage as long as all the other firms did so in the past stages and whether one of the firms deviates from the collusive path the whole industry reverts to the competitive equilibrium forever. Therefore collusion is only sustained in equilibrium if the long run gains of the cartel exceed the one shot gain from deviation, that is, if the following incentive compatibility constrain (ICC) is verified:

$$\left(\pi_i^C - \pi_i^P\right)\frac{\delta}{1-\delta} \geq \pi_i^D - \pi_i^C, \tag{B.1}$$

where π_i^C is the share of the profit of the cartel received by firm i, π_i^P is the profit of the punishment phase (equal to the competitive profit), π_i^D is the profit earned by firm i when it is the only one deviating from the collusive path and δ is the discount factor. The left-hand side of the ICC is the present value of all future gains of the cartel and the right-hand side of the equation is the one shot gain from deviation. Isolating the discount factor in equation (B.1) allows us to rewrite the incentive compatibility constraint as:

$$\delta \geq \frac{\pi_i^D - \pi_i^C}{\pi_i^D - \pi_i^P}, \qquad i = 1, \dots, n. \tag{B.2}$$

From equation (B.2) it is possible to conclude that if every firm earns strictly more under collusion than under competition ($\pi_i^C > \pi_i^P$), there are discount factors lower than one for which informal collusion can be sustained in equilibrium.

Theorem: under optimal penal codes there are always discount rates lower than one for which collusion can be sustained in equilibrium at every stage of an infinite super game.

Proof: In order to be a subgame-perfect equilibrium, optimal penal codes must verify two incentive compatibility constraints. One the one hand, in the collusive subgames all firms must prefer to collude rather than to deviate and being punished afterwards:

$$\pi_i^C \frac{1}{1-\delta} \geq \pi_i^D + \delta V, \tag{B.3}$$

where V is the present discounted value of the profits received in the punishment phase. On the other hand, all firms must prefer to accept the punishment rather than to deviate and have the punishment restarted. The ICC of the punishment subgames is thus given by:

$$V \geq \pi_i^{DP} + \delta V. \tag{B.4}$$

π_i^{DP} is the profit earned when firm i deviates from the punishment phase. Equation (B.4) holds when the punishment is extremely severe in the first stage ("stick") while in the following stages firms are rewarded by returning to the collusive path ("carrot").

The penal code is optimal when the present value of the profits earned in the punishment phase is as low as possible. Because firms can leave the market to avoid future losses, at the optimal (security level) penal code V is equal to zero and equation (B.3) is equivalent to:

$$\delta \geq \frac{\pi_i^D - \pi_i^C}{\pi_i^D}. \tag{B.5}$$

At last, from equation (B.5) we conclude that under optimal penal codes there are discount factors lower than one for which collusion can be sustained.

We can further conclude by comparing equations (B.2) and (B.5) that the use of an optimal penal code instead of a trigger strategy also increases the set of discount rates for which collusion is possible.

Appendix C

In this appendix we do comparative statics to prove that a positive variation in the wage that firm i offers to specialized workers has, in average, a negative impact on the price of the same firm and a positive or negative impact on the prices of the other firms, depending on the characteristics of the industry. In mathematical notation:

$$\frac{\partial P_i(.)}{\partial W_i} < 0 \qquad and \qquad \begin{cases} \dfrac{\partial P_k(.)}{\partial W_i} < 0, & if\ industry = type\ A \\ \dfrac{\partial P_k(.)}{\partial W_i} > 0, & if\ industry = type\ B \end{cases}, \qquad k \neq i.$$

Along our proof we assume symmetry, because this result may not hold for every firm when companies are too different. To understand why we must use comparative statics to compute the two derivatives, remark that when firm i changes the wage paid to specialized workers in the first stage, this completely alters the equilibrium played in the second stage, in which firms choose different prices and levels of non specialized labor. Nevertheless, regardless of the wage fixed, firm i optimizes profits in the second stage and so the first order conditions given by equations (3.3.4) and (3.3.5) are always verified. In other words, the two following functions $A_i\left(P_1, \dots, P_n, L_{2_i}, W_i\right)$ and $B_i\left(P_1, \dots, P_n, L_{2_i}, W_i\right)$ remain constant and equal to zero, because any change in W_i is offset with changes in prices and L_{2_i}:

$$A_i\left(P_1, \dots, P_n, L_{2_i}, W_i\right) = f_i(P_1, \dots, P_n) + \left(P_i - \overline{W}\,\frac{\partial L_{2_i}}{\partial h_i}\right)\frac{\partial f_i(P_1, \dots, P_n)}{\partial P_i} = 0.$$

$$B_i\left(P_1, \dots, P_n, L_{2_i}, W_i\right) = f_i(P_1, \dots, P_n) - h_i\left(g_i(W_1, \dots, W_n), L_{2_i}\right) = 0.$$

Using a generalization of the implicit function theorem for three variables, we can prove that:

$$\begin{cases} \dfrac{\partial A_i}{\partial P_1} dP_1 + \cdots + \dfrac{\partial A_i}{\partial P_n} dP_n + \dfrac{\partial A_i}{\partial L_{2_i}} dL_{2_i} + \dfrac{\partial A_i}{\partial W_i} dW_i = 0 \\ \dfrac{\partial B_i}{\partial P_1} dP_1 + \cdots + \dfrac{\partial B_i}{\partial P_n} dP_n + \dfrac{\partial B_i}{\partial L_{2_i}} dL_{2_i} + \dfrac{\partial B_i}{\partial W_i} dW_i = 0 \end{cases}.$$

Under symmetry:

$$\begin{cases} \frac{\partial A_i}{\partial P_i} dP_i + (n-1) \frac{\partial A_i}{\partial P_k} dP_k + \frac{\partial A_i}{\partial L_{2i}} dL_{2i} + \frac{\partial A_i}{\partial W_i} dW_i = 0 \\ \frac{\partial B_i}{\partial P_i} dP_i + (n-1) \frac{\partial B_i}{\partial P_k} dP_k + \frac{\partial B_i}{\partial L_{2i}} dL_{2i} + \frac{\partial B_i}{\partial W_i} dW_i = 0 \end{cases}.$$

Dividing every term by dW_i gives

$$\begin{cases} \frac{\partial A_i}{\partial P_i} \frac{dP_i}{dW_i} + (n-1) \frac{\partial A_i}{\partial P_k} \frac{dP_k}{dW_i} + \frac{\partial A_i}{\partial L_{2i}} \frac{dL_{2i}}{dW_i} + \frac{\partial A_i}{\partial W_i} = 0 \\ \frac{\partial B_i}{\partial P_i} \frac{dP_i}{dW_i} + (n-1) \frac{\partial B_i}{\partial P_k} \frac{dP_k}{dW_i} + \frac{\partial B_i}{\partial L_{2i}} \frac{dL_{2i}}{dW_i} + \frac{\partial B_i}{\partial W_i} = 0 \end{cases}.$$

Solving for $\frac{dL_{2i}}{dW_i}$:

$$\begin{cases} \frac{dL_{2i}}{dW_i} = -\frac{\frac{\partial A_i}{\partial W_i} + \frac{\partial A_i}{\partial P_i}\frac{dP_i}{dW_i} + (n-1)\frac{\partial A_i}{\partial P_k}\frac{dP_k}{dW_i}}{\frac{\partial A_i}{\partial L_{2i}}} \\ \frac{dL_{2i}}{dW_i} = -\frac{\frac{\partial B_i}{\partial W_i} + \frac{\partial B_i}{\partial P_i}\frac{dP_i}{dW_i} + (n-1)\frac{\partial B_i}{\partial P_k}\frac{dP_k}{dW_i}}{\frac{\partial B_i}{\partial L_{2i}}} \end{cases}.$$

Equating the right-handed side of the two previous equations to eliminate $\frac{dL_{2i}}{dW_i}$ gives:

$$(n-1)\left(\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2i}} - \frac{\partial A_i}{\partial L_{2i}}\frac{\partial B_i}{\partial P_k}\right)\frac{dP_k}{dW_i} + \left(\frac{\partial A_i}{\partial P_i}\frac{\partial B_i}{\partial L_{2i}} - \frac{\partial A_i}{\partial L_{2i}}\frac{\partial B_i}{\partial P_i}\right)\frac{dP_i}{dW_i} =$$

$$= \frac{\partial A_i}{\partial L_{2i}}\frac{\partial B_i}{\partial W_i} - \frac{\partial A_i}{\partial W_i}\frac{\partial B_i}{\partial L_{2i}}. \qquad (\text{C}.1)$$

In the same way that firm i always optimizes individual profits in the second stage of the game for any W_i, the other firms of the industry do the same. Then the functions $A_k(P_1, \dots, P_n, L_{2_k}, W_i)$ and $B_k(P_1, \dots, P_n, L_{2_k}, W_i)$ also remain constant and equal to zero when the wage paid by firm i is changed.

$$A_k(P_1, \dots, P_n, L_{2_k}, W_i) = f_k(P_1, \dots, P_n) + \left(P_k - \bar{W}\frac{\partial L_{2k}}{\partial h_k}\right)\frac{\partial f_k(P_1, \dots, P_n)}{\partial P_k} = 0.$$

$$B_k(P_1, \dots, P_n, L_{2_k}, W_i) = f_k(P_1, \dots, P_n) - h_k\left(g_k(W_1, \dots, W_n), L_{2_k}\right) = 0.$$

Using the same reasoning as before:

$$\begin{cases} \dfrac{\partial A_k}{\partial P_1}dP_1 + \dots + \dfrac{\partial A_k}{\partial P_n}dP_n + \dfrac{\partial A_k}{\partial L_{2_k}}dL_{2k} + \dfrac{\partial A_k}{\partial W_i}dW_i = 0 \\ \dfrac{\partial B_k}{\partial P_1}dP_1 + \dots + \dfrac{\partial B_k}{\partial P_n}dP_n + \dfrac{\partial B_k}{\partial L_{2_k}}dL_{2k} + \dfrac{\partial B_k}{\partial W_i}dW_i = 0 \end{cases}.$$

Under symmetry,

$$\begin{cases} (n-2)\dfrac{\partial A_k}{\partial P_i}dP_k + \dfrac{\partial A_k}{\partial P_k}dP_k + \dfrac{\partial A_k}{\partial P_i}dP_i + \dfrac{\partial A_k}{\partial L_{2_k}}dL_{2k} + \dfrac{\partial A_k}{\partial W_i}dW_i = 0 \\ (n-2)\dfrac{\partial B_k}{\partial P_i}dP_k + \dfrac{\partial B_k}{\partial P_k}dP_k + \dfrac{\partial B_k}{\partial P_i}dP_i + \dfrac{\partial B_k}{\partial L_{2_k}}dL_{2k} + \dfrac{\partial B_k}{\partial W_i}dW_i = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \quad \begin{cases} \left((n-2)\dfrac{\partial A_k}{\partial P_i} + \dfrac{\partial A_k}{\partial P_k}\right)dP_k + \dfrac{\partial A_k}{\partial P_i}dP_i + \dfrac{\partial A_k}{\partial L_{2_k}}dL_{2k} + \dfrac{\partial A_k}{\partial W_i}dW_i = 0 \\ \left((n-2)\dfrac{\partial B_k}{\partial P_i} + \dfrac{\partial B_k}{\partial P_k}\right)dP_k + \dfrac{\partial B_k}{\partial P_i}dP_i + \dfrac{\partial B_k}{\partial L_{2_k}}dL_{2k} + \dfrac{\partial B_k}{\partial W_i}dW_i = 0 \end{cases}.$$

Dividing every term by dW_i and solving for $\frac{dL_{2k}}{dW_i}$ gives:

$$\begin{cases} \left((n-2)\dfrac{\partial A_k}{\partial P_i} + \dfrac{\partial A_k}{\partial P_k}\right)\dfrac{dP_k}{dW_i} + \dfrac{\partial A_k}{\partial P_i}\dfrac{dP_i}{dW_i} + \dfrac{\partial A_k}{\partial L_{2_k}}\dfrac{dL_{2k}}{dW_i} + \dfrac{\partial A_k}{\partial W_i} = 0 \\ \left((n-2)\dfrac{\partial B_k}{\partial P_i} + \dfrac{\partial B_k}{\partial P_k}\right)\dfrac{dP_k}{dW_i} + \dfrac{\partial B_k}{\partial P_i}\dfrac{dP_i}{dW_i} + \dfrac{\partial B_k}{\partial L_{2_k}}\dfrac{dL_{2k}}{dW_i} + \dfrac{\partial B_k}{\partial W_i} = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \dfrac{dL_{2_k}}{dW_i} = -\dfrac{\dfrac{\partial A_k}{\partial W_i} + \left((n-2)\dfrac{\partial A_k}{\partial P_i} + \dfrac{\partial A_k}{\partial P_k}\right)\dfrac{dP_k}{dW_i} + \dfrac{\partial A_k}{\partial P_i}\dfrac{dP_i}{dW_i}}{\dfrac{\partial A_k}{\partial L_{2_k}}} \\ \dfrac{dL_{2_k}}{dW_i} = -\dfrac{\dfrac{\partial B_k}{\partial W_i} + \left((n-2)\dfrac{\partial B_k}{\partial P_i} + \dfrac{\partial B_k}{\partial P_k}\right)\dfrac{dP_k}{dW_i} + \dfrac{\partial B_k}{\partial P_i}\dfrac{dP_i}{dW_i}}{\dfrac{\partial B_k}{\partial L_{2_k}}} \end{cases}.$$

Once again we eliminate $\frac{dL_{2_k}}{dW_i}$ and get another equation relating $\frac{dP_i}{dW_i}$ and $\frac{dP_k}{dW_i}$:

$$\left((n-2)\frac{\partial A_k}{\partial P_i}\frac{\partial B_k}{\partial L_{2_k}} + \frac{\partial A_k}{\partial P_k}\frac{\partial B_k}{\partial L_{2_k}} - (n-2)\frac{\partial A_k}{\partial L_{2_k}}\frac{\partial B_k}{\partial P_i} - \frac{\partial A_k}{\partial L_{2_k}}\frac{\partial B_k}{\partial P_k}\right)\frac{dP_k}{dW_i} +$$

$$+\left(\frac{\partial A_k}{\partial P_i}\frac{\partial B_k}{\partial L_{2_k}} - \frac{\partial A_k}{\partial L_{2_k}}\frac{\partial B_k}{\partial P_i}\right)\frac{dP_i}{dW_i} = \frac{\partial A_k}{\partial L_{2_k}}\frac{\partial B_k}{\partial W_i} - \frac{\partial A_k}{\partial W_i}\frac{\partial B_k}{\partial L_{2_k}}. \qquad (C.2)$$

The system of equations (C.1) and (C.2) can now be used to find $\frac{dP_i}{dW_i}$ and $\frac{dP_k}{dW_i}$. Solving equation (C.1) for $\frac{dP_k}{dW_i}$:

$$\frac{dP_k}{dW_i} = \frac{\dfrac{\partial A_i}{\partial L_{2_i}}\dfrac{\partial B_i}{\partial W_i} - \dfrac{\partial A_i}{\partial W_i}\dfrac{\partial B_i}{\partial L_{2_i}} + \left(\dfrac{\partial A_i}{\partial L_{2_i}}\dfrac{\partial B_i}{\partial P_i} - \dfrac{\partial A_i}{\partial P_i}\dfrac{\partial B_i}{\partial L_{2_i}}\right)\dfrac{dP_i}{dW_i}}{(n-1)\dfrac{\partial A_i}{\partial P_k}\dfrac{\partial B_i}{\partial L_{2_i}} - (n-1)\dfrac{\partial A_i}{\partial L_{2_i}}\dfrac{\partial B_i}{\partial P_k}}.$$

Solving equation (C.2) for $\frac{dP_k}{dW_i}$ and applying symmetry:

$$\frac{dP_k}{dW_i} = \frac{\dfrac{\partial A_i}{\partial L_{2_i}}\dfrac{\partial B_i}{\partial W_k} - \dfrac{\partial A_i}{\partial W_k}\dfrac{\partial B_i}{\partial L_{2_i}} + \left(\dfrac{\partial A_i}{\partial L_{2_i}}\dfrac{\partial B_i}{\partial P_k} - \dfrac{\partial A_i}{\partial P_k}\dfrac{\partial B_i}{\partial L_{2_i}}\right)\dfrac{dP_i}{dW_i}}{(n-2)\dfrac{\partial A_i}{\partial P_k}\dfrac{\partial B_i}{\partial L_{2_i}} + \dfrac{\partial A_i}{\partial P_i}\dfrac{\partial B_i}{\partial L_{2_i}} - (n-2)\dfrac{\partial A_i}{\partial L_{2_i}}\dfrac{\partial B_i}{\partial P_k} - \dfrac{\partial A_i}{\partial L_{2_i}}\dfrac{\partial B_i}{\partial P_i}}.$$

Equating the two previous equations:

$$\left(\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k}-\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}}\right)\left((n-1)\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}}-(n-1)\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k}\right)\frac{dP_i}{dW_i}-$$

$$-\left((n-2)\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}}+\frac{\partial A_i}{\partial P_i}\frac{\partial B_i}{\partial L_{2_i}}-(n-2)\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k}-\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_i}\right)\left(\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_i}-\frac{\partial A_i}{\partial P_i}\frac{\partial B_i}{\partial L_{2_i}}\right)\frac{dP_i}{dW_i}$$

$$=\left(\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial W_i}-\frac{\partial A_i}{\partial W_i}\frac{\partial B_i}{\partial L_{2_i}}\right)\left((n-2)\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}}+\frac{\partial A_i}{\partial P_i}\frac{\partial B_i}{\partial L_{2_i}}-(n-2)\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k}-\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_i}\right)$$

$$-\left(\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial W_k}-\frac{\partial A_i}{\partial W_k}\frac{\partial B_i}{\partial L_{2_i}}\right)\left((n-1)\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}}-(n-1)\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k}\right)$$

$$\Leftrightarrow$$

$$\left(\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k}-\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}}-\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_i}+\frac{\partial A_i}{\partial P_i}\frac{\partial B_i}{\partial L_{2_i}}\right)\left((n-1)\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}}+\frac{\partial A_i}{\partial P_i}\frac{\partial B_i}{\partial L_{2_i}}\right.$$

$$\left.-(n-1)\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k}-\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_i}\right)\frac{dP_i}{dW_i}=$$

$$=\left(\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial W_i}-\frac{\partial A_i}{\partial W_i}\frac{\partial B_i}{\partial L_{2_i}}-\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial W_k}+\frac{\partial A_i}{\partial W_k}\frac{\partial B_i}{\partial L_{2_i}}\right)\left((n-1)\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}}-(n-1)\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k}\right)$$

$$+\left(\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial W_i}-\frac{\partial A_i}{\partial W_i}\frac{\partial B_i}{\partial L_{2_i}}\right)\left(\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k}-\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}}-\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_i}+\frac{\partial A_i}{\partial P_i}\frac{\partial B_i}{\partial L_{2_i}}\right).$$

Solving for $\frac{dP_i}{dW_i}$:

$$\frac{dP_i}{dW_i}=-\frac{\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial W_k}-\frac{\partial A_i}{\partial W_k}\frac{\partial B_i}{\partial L_{2_i}}-\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial W_i}+\frac{\partial A_i}{\partial W_i}\frac{\partial B_i}{\partial L_{2_i}}}{\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k}-\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}}-\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_i}+\frac{\partial A_i}{\partial P_i}\frac{\partial B_i}{\partial L_{2_i}}}\times$$

$$\times\frac{(n-1)\left(\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}}-\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k}\right)}{(n-1)\left(\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}}-\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k}\right)+\frac{\partial A_i}{\partial P_i}\frac{\partial B_i}{\partial L_{2_i}}-\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_i}}+$$

$$+\frac{\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial W_i}-\frac{\partial A_i}{\partial W_i}\frac{\partial B_i}{\partial L_{2_i}}}{(n-1)\left(\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}}-\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k}\right)+\frac{\partial A_i}{\partial P_i}\frac{\partial B_i}{\partial L_{2_i}}-\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_i}}.$$

It is possible to simplify this expression by factoring out the common factor:

$$\frac{dP_i}{dW_i} = \frac{1}{(n-1)\left(\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}} - \frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k}\right) + \frac{\partial A_i}{\partial P_i}\frac{\partial B_i}{\partial L_{2_i}} - \frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_i}} \times$$

$$\times\left[-\frac{\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial W_k} - \frac{\partial A_i}{\partial W_k}\frac{\partial B_i}{\partial L_{2_i}} - \frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial W_i} + \frac{\partial A_i}{\partial W_i}\frac{\partial B_i}{\partial L_{2_i}}}{\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k} - \frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}} - \frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_i} + \frac{\partial A_i}{\partial P_i}\frac{\partial B_i}{\partial L_{2_i}}} \times (n-1)\left(\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}} - \frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k}\right) + \right.$$

$$\left. + \frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial W_i} - \frac{\partial A_i}{\partial W_i}\frac{\partial B_i}{\partial L_{2_i}}\right].$$

Dividing out now the factor $\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k} - \frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}} - \frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_i} + \frac{\partial A_i}{\partial P_i}\frac{\partial B_i}{\partial L_{2_i}}$ from the terms inside the straight parentheses:

$$\frac{dP_i}{dW_i} = \frac{1}{(n-1)\left(\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}} - \frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k}\right) + \frac{\partial A_i}{\partial P_i}\frac{\partial B_i}{\partial L_{2_i}} - \frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_i}} \times$$

$$\times \frac{1}{\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k} - \frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}} - \frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_i} + \frac{\partial A_i}{\partial P_i}\frac{\partial B_i}{\partial L_{2_i}}} \times$$

$$\times\left[(n-1)\left(-\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial W_k} + \frac{\partial A_i}{\partial W_k}\frac{\partial B_i}{\partial L_{2_i}} + \frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial W_i} - \frac{\partial A_i}{\partial W_i}\frac{\partial B_i}{\partial L_{2_i}}\right)\left(\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}} - \frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k}\right) + \right.$$

$$\left. + \left(\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial W_i} - \frac{\partial A_i}{\partial W_i}\frac{\partial B_i}{\partial L_{2_i}}\right)\left(\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k} - \frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}} - \frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_i} + \frac{\partial A_i}{\partial P_i}\frac{\partial B_i}{\partial L_{2_i}}\right)\right]$$

$$\Leftrightarrow$$

$$\frac{dP_i}{dW_i} = \frac{1}{(n-1)\left(\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}} - \frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k}\right) + \frac{\partial A_i}{\partial P_i}\frac{\partial B_i}{\partial L_{2_i}} - \frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_i}} \times$$

$$\times \frac{1}{\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k} - \frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}} - \frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_i} + \frac{\partial A_i}{\partial P_i}\frac{\partial B_i}{\partial L_{2_i}}} \times$$

$$\Bigg[-(n-1)\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial W_k}\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}} + (n-1)\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial W_k}\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k} +$$

$$+(n-1)\frac{\partial A_i}{\partial W_k}\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}} - (n-1)\frac{\partial A_i}{\partial W_k}\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k} +$$

$$+(n-1)\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial W_i}\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}} - (n-1)\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial W_i}\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k} -$$

$$-(n-1)\frac{\partial A_i}{\partial W_i}\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}} + (n-1)\frac{\partial A_i}{\partial W_i}\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k} +$$

$$+\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial W_i}\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k} - \frac{\partial A_i}{\partial W_i}\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k} - \frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial W_i}\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}} + \frac{\partial A_i}{\partial W_i}\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}} -$$

$$-\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial W_i}\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_i} + \frac{\partial A_i}{\partial W_i}\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_i} + \frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial W_i}\frac{\partial A_i}{\partial P_i}\frac{\partial B_i}{\partial L_{2_i}} - \frac{\partial A_i}{\partial W_i}\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial P_i}\frac{\partial B_i}{\partial L_{2_i}}\Bigg].$$

Rearranging the terms in brackets:

$$\frac{dP_i}{dW_i} = \frac{1}{\frac{\partial B_i}{\partial L_{2_i}}\left(\frac{\partial A_i}{\partial P_i} + (n-1)\frac{\partial A_i}{\partial P_k}\right) - \frac{\partial A_i}{\partial L_{2_i}}\left(\frac{\partial B_i}{\partial P_i} + (n-1)\frac{\partial B_i}{\partial P_k}\right)} \times$$

$$\times \frac{1}{\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k} - \frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}} - \frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_i} + \frac{\partial A_i}{\partial P_i}\frac{\partial B_i}{\partial L_{2_i}}} \times$$

$$\times\Bigg[-\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}}\left(\frac{\partial B_i}{\partial W_i} + (n-1)\frac{\partial B_i}{\partial W_k}\right) + \frac{\partial A_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k}\left(\frac{\partial B_i}{\partial W_i} + (n-1)\frac{\partial B_i}{\partial W_k}\right) +$$

$$+\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}}\left(\frac{\partial A_i}{\partial W_i} + (n-1)\frac{\partial A_i}{\partial W_k}\right) - \frac{\partial B_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k}\left(\frac{\partial A_i}{\partial W_i} + (n-1)\frac{\partial A_i}{\partial W_k}\right) +$$

$$+\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial W_i}\frac{\partial B_i}{\partial L_{2_i}}\left(\frac{\partial A_i}{\partial P_i} + (n-1)\frac{\partial A_i}{\partial P_k}\right) - \frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial W_i}\frac{\partial A_i}{\partial L_{2_i}}\left(\frac{\partial B_i}{\partial P_i} + (n-1)\frac{\partial B_i}{\partial P_k}\right) -$$

$$-\frac{\partial A_i}{\partial W_i}\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial L_{2_i}}\left(\frac{\partial A_i}{\partial P_i} + (n-1)\frac{\partial A_i}{\partial P_k}\right) + \frac{\partial A_i}{\partial W_i}\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial L_{2_i}}\left(\frac{\partial B_i}{\partial P_i} + (n-1)\frac{\partial B_i}{\partial P_k}\right)\Bigg] \qquad (C)$$

From the signs of the partial derivatives of functions A_i and B_i given in Table 9 of Appendix D, it follows that the right handed side of equation (C.3) is negative and so we conclude that:

$$\frac{dP_i}{dW_i} < 0.$$

To get a simplified expression for $\frac{dP_k}{dW_i}$ we start by subtracting the terms in equation (C.1) to the terms in equation (C.2):

$$\left((n-2)\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}} + \frac{\partial A_i}{\partial P_i}\frac{\partial B_i}{\partial L_{2_i}} - (n-2)\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k} - \frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_i}\right)\frac{dP_k}{dW_i} -$$

$$-(n-1)\left(\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}} - \frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k}\right)\frac{dP_k}{dW_i} +$$

$$+\left(\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}} - \frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k}\right)\frac{dP_i}{dW_i} - \left(\frac{\partial A_i}{\partial P_i}\frac{\partial B_i}{\partial L_{2_i}} - \frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_i}\right)\frac{dP_i}{dW_i} =$$

$$= \frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial W_k} - \frac{\partial A_i}{\partial W_k}\frac{\partial B_i}{\partial L_{2_i}} - \frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial W_i} + \frac{\partial A_i}{\partial W_i}\frac{\partial B_i}{\partial L_{2_i}}$$

$$\Leftrightarrow$$

$$\left(\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k} - \frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}} - \frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_i} + \frac{\partial A_i}{\partial P_i}\frac{\partial B_i}{\partial L_{2_i}}\right)\frac{dP_k}{dW_i}$$

$$-\left(\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k} - \frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}} - \frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_i} + \frac{\partial A_i}{\partial P_i}\frac{\partial B_i}{\partial L_{2_i}}\right)\frac{dP_i}{dW_i} =$$

$$= \frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial W_k} - \frac{\partial A_i}{\partial W_k}\frac{\partial B_i}{\partial L_{2_i}} - \frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial W_i} + \frac{\partial A_i}{\partial W_i}\frac{\partial B_i}{\partial L_{2_i}}.$$

Solving for $\frac{dP_k}{dW_i}$:

$$\frac{dP_k}{dW_i} = \frac{dP_i}{dW_i} + \frac{\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial W_k} - \frac{\partial A_i}{\partial W_k}\frac{\partial B_i}{\partial L_{2_i}} - \frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial W_i} + \frac{\partial A_i}{\partial W_i}\frac{\partial B_i}{\partial L_{2_i}}}{\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k} - \frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}} - \frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_i} + \frac{\partial A_i}{\partial P_i}\frac{\partial B_i}{\partial L_{2_i}}}.$$

Replacing $\frac{dP_i}{dW_i}$ by equation (C.3) and factoring out the two coefficients gives:

$$\frac{dP_k}{dW_i} = \frac{1}{\frac{\partial B_i}{\partial L_{2_i}}\left(\frac{\partial A_i}{\partial P_i} + (n-1)\frac{\partial A_i}{\partial P_k}\right) - \frac{\partial A_i}{\partial L_{2_i}}\left(\frac{\partial B_i}{\partial P_i} + (n-1)\frac{\partial B_i}{\partial P_k}\right)} \times$$

$$\times \frac{1}{\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k} - \frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}} - \frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_i} + \frac{\partial A_i}{\partial P_i}\frac{\partial B_i}{\partial L_{2_i}}} \times$$

$$\times \left[-\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}}\left(\frac{\partial B_i}{\partial W_i} + (n-1)\frac{\partial B_i}{\partial W_k}\right) + \frac{\partial A_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k}\left(\frac{\partial B_i}{\partial W_i} + (n-1)\frac{\partial B_i}{\partial W_k}\right) + \right.$$

$$+ \frac{\partial B_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}}\left(\frac{\partial A_i}{\partial W_i} + (n-1)\frac{\partial A_i}{\partial W_k}\right) - \frac{\partial B_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k}\left(\frac{\partial A_i}{\partial W_i} + (n-1)\frac{\partial A_i}{\partial W_k}\right) +$$

$$+ \frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial W_i}\frac{\partial B_i}{\partial L_{2_i}}\left(\frac{\partial A_i}{\partial P_i} + (n-1)\frac{\partial A_i}{\partial P_k}\right) - \frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial W_i}\frac{\partial A_i}{\partial L_{2_i}}\left(\frac{\partial B_i}{\partial P_i} + (n-1)\frac{\partial B_i}{\partial P_k}\right) -$$

$$- \frac{\partial A_i}{\partial W_i}\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial L_{2_i}}\left(\frac{\partial A_i}{\partial P_i} + (n-1)\frac{\partial A_i}{\partial P_k}\right) + \frac{\partial A_i}{\partial W_i}\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial L_{2_i}}\left(\frac{\partial B_i}{\partial P_i} + (n-1)\frac{\partial B_i}{\partial P_k}\right) +$$

$$+ \left(\frac{\partial B_i}{\partial L_{2_i}}\left(\frac{\partial A_i}{\partial P_i} + (n-1)\frac{\partial A_i}{\partial P_k}\right) - \frac{\partial A_i}{\partial L_{2_i}}\left(\frac{\partial B_i}{\partial P_i} + (n-1)\frac{\partial B_i}{\partial P_k}\right)\right) \times$$

$$\left. \times \left(\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial W_k} - \frac{\partial A_i}{\partial W_k}\frac{\partial B_i}{\partial L_{2_i}} - \frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial W_i} + \frac{\partial A_i}{\partial W_i}\frac{\partial B_i}{\partial L_{2_i}}\right)\right].$$

Rearranging the two last terms:

$$\frac{dP_k}{dW_i} = \frac{1}{\frac{\partial B_i}{\partial L_{2_i}}\left(\frac{\partial A_i}{\partial P_i} + (n-1)\frac{\partial A_i}{\partial P_k}\right) - \frac{\partial A_i}{\partial L_{2_i}}\left(\frac{\partial B_i}{\partial P_i} + (n-1)\frac{\partial B_i}{\partial P_k}\right)} \times$$

$$\times \frac{1}{\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k} - \frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}} - \frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_i} + \frac{\partial A_i}{\partial P_i}\frac{\partial B_i}{\partial L_{2_i}}} \times$$

$$\times \left[-\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}}\left(\frac{\partial B_i}{\partial W_i} + (n-1)\frac{\partial B_i}{\partial W_k}\right) + \frac{\partial A_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k}\left(\frac{\partial B_i}{\partial W_i} + (n-1)\frac{\partial B_i}{\partial W_k}\right) + \right.$$

$$+\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}}\left(\frac{\partial A_i}{\partial W_i} + (n-1)\frac{\partial A_i}{\partial W_k}\right) - \frac{\partial B_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k}\left(\frac{\partial A_i}{\partial W_i} + (n-1)\frac{\partial A_i}{\partial W_k}\right) +$$

$$+\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial W_i}\frac{\partial B_i}{\partial L_{2_i}}\left(\frac{\partial A_i}{\partial P_i} + (n-1)\frac{\partial A_i}{\partial P_k}\right) - \frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial W_i}\frac{\partial A_i}{\partial L_{2_i}}\left(\frac{\partial B_i}{\partial P_i} + (n-1)\frac{\partial B_i}{\partial P_k}\right) -$$

$$-\frac{\partial A_i}{\partial W_i}\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial L_{2_i}}\left(\frac{\partial A_i}{\partial P_i} + (n-1)\frac{\partial A_i}{\partial P_k}\right) + \frac{\partial A_i}{\partial W_i}\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial L_{2_i}}\left(\frac{\partial B_i}{\partial P_i} + (n-1)\frac{\partial B_i}{\partial P_k}\right) -$$

$$+\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial W_k}\frac{\partial B_i}{\partial L_{2_i}}\left(\frac{\partial A_i}{\partial P_i} + (n-1)\frac{\partial A_i}{\partial P_k}\right) - \frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial W_k}\frac{\partial A_i}{\partial L_{2_i}}\left(\frac{\partial B_i}{\partial P_i} + (n-1)\frac{\partial B_i}{\partial P_k}\right) -$$

$$-\frac{\partial A_i}{\partial W_k}\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial L_{2_i}}\left(\frac{\partial A_i}{\partial P_i} + (n-1)\frac{\partial A_i}{\partial P_k}\right) + \frac{\partial A_i}{\partial W_k}\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial L_{2_i}}\left(\frac{\partial B_i}{\partial P_i} + (n-1)\frac{\partial B_i}{\partial P_k}\right) -$$

$$-\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial W_i}\frac{\partial B_i}{\partial L_{2_i}}\left(\frac{\partial A_i}{\partial P_i} + (n-1)\frac{\partial A_i}{\partial P_k}\right) + \frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial W_i}\frac{\partial A_i}{\partial L_{2_i}}\left(\frac{\partial B_i}{\partial P_i} + (n-1)\frac{\partial B_i}{\partial P_k}\right) +$$

$$\left. +\frac{\partial A_i}{\partial W_i}\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial L_{2_i}}\left(\frac{\partial A_i}{\partial P_i} + (n-1)\frac{\partial A_i}{\partial P_k}\right) - \frac{\partial A_i}{\partial W_i}\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial L_{2_i}}\left(\frac{\partial B_i}{\partial P_i} + (n-1)\frac{\partial B_i}{\partial P_k}\right)\right].$$

Eliminating the repeated terms in brackets:

$$\frac{dP_k}{dW_i} = \frac{1}{\frac{\partial B_i}{\partial L_{2i}}\left(\frac{\partial A_i}{\partial P_i} + (n-1)\frac{\partial A_i}{\partial P_k}\right) - \frac{\partial A_i}{\partial L_{2i}}\left(\frac{\partial B_i}{\partial P_i} + (n-1)\frac{\partial B_i}{\partial P_k}\right)} \times$$

$$\times \frac{1}{\frac{\partial A_i}{\partial L_{2i}}\frac{\partial B_i}{\partial P_k} - \frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2i}} - \frac{\partial A_i}{\partial L_{2i}}\frac{\partial B_i}{\partial P_i} + \frac{\partial A_i}{\partial P_i}\frac{\partial B_i}{\partial L_{2i}}} \times$$

$$\times \left[-\frac{\partial A_i}{\partial L_{2i}}\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2i}}\left(\frac{\partial B_i}{\partial W_i} + (n-1)\frac{\partial B_i}{\partial W_k}\right) + \frac{\partial A_i}{\partial L_{2i}}\frac{\partial A_i}{\partial L_{2i}}\frac{\partial B_i}{\partial P_k}\left(\frac{\partial B_i}{\partial W_i} + (n-1)\frac{\partial B_i}{\partial W_k}\right) + \right.$$

$$+ \frac{\partial B_i}{\partial L_{2i}}\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2i}}\left(\frac{\partial A_i}{\partial W_i} + (n-1)\frac{\partial A_i}{\partial W_k}\right) - \frac{\partial B_i}{\partial L_{2i}}\frac{\partial A_i}{\partial L_{2i}}\frac{\partial B_i}{\partial P_k}\left(\frac{\partial A_i}{\partial W_i} + (n-1)\frac{\partial A_i}{\partial W_k}\right) +$$

$$+ \frac{\partial A_i}{\partial L_{2i}}\frac{\partial B_i}{\partial W_k}\frac{\partial B_i}{\partial L_{2i}}\left(\frac{\partial A_i}{\partial P_i} + (n-1)\frac{\partial A_i}{\partial P_k}\right) - \frac{\partial A_i}{\partial L_{2i}}\frac{\partial B_i}{\partial W_k}\frac{\partial A_i}{\partial L_{2i}}\left(\frac{\partial B_i}{\partial P_i} + (n-1)\frac{\partial B_i}{\partial P_k}\right) -$$

$$\left. -\frac{\partial A_i}{\partial W_k}\frac{\partial B_i}{\partial L_{2i}}\frac{\partial B_i}{\partial L_{2i}}\left(\frac{\partial A_i}{\partial P_i} + (n-1)\frac{\partial A_i}{\partial P_k}\right) + \frac{\partial A_i}{\partial W_k}\frac{\partial B_i}{\partial L_{2i}}\frac{\partial A_i}{\partial L_{2i}}\left(\frac{\partial B_i}{\partial P_i} + (n-1)\frac{\partial B_i}{\partial P_k}\right)\right]$$

$$\Leftrightarrow$$

$$\frac{dP_k}{dW_i} = \frac{1}{\frac{\partial B_i}{\partial L_{2i}}\left(\frac{\partial A_i}{\partial P_i} + (n-1)\frac{\partial A_i}{\partial P_k}\right) - \frac{\partial A_i}{\partial L_{2i}}\left(\frac{\partial B_i}{\partial P_i} + (n-1)\frac{\partial B_i}{\partial P_k}\right)} \times$$

$$\times \frac{1}{\frac{\partial A_i}{\partial L_{2i}}\frac{\partial B_i}{\partial P_k} - \frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2i}} - \frac{\partial A_i}{\partial L_{2i}}\frac{\partial B_i}{\partial P_i} + \frac{\partial A_i}{\partial P_i}\frac{\partial B_i}{\partial L_{2i}}} \times$$

$$\times \left[-\frac{\partial A_i}{\partial L_{2i}}\frac{\partial B_i}{\partial L_{2i}}\left(\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial W_i} + (n-1)\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial W_k}\right) + \frac{\partial A_i}{\partial L_{2i}}\frac{\partial A_i}{\partial L_{2i}}\left(\frac{\partial B_i}{\partial P_k}\frac{\partial B_i}{\partial W_i} + (n-1)\frac{\partial B_i}{\partial P_k}\frac{\partial B_i}{\partial W_k}\right) + \right.$$

$$+ \frac{\partial B_i}{\partial L_{2i}}\frac{\partial B_i}{\partial L_{2i}}\left(\frac{\partial A_i}{\partial P_k}\frac{\partial A_i}{\partial W_i} + (n-1)\frac{\partial A_i}{\partial P_k}\frac{\partial A_i}{\partial W_k}\right) - \frac{\partial B_i}{\partial L_{2i}}\frac{\partial A_i}{\partial L_{2i}}\left(\frac{\partial B_i}{\partial P_k}\frac{\partial A_i}{\partial W_i} + (n-1)\frac{\partial B_i}{\partial P_k}\frac{\partial A_i}{\partial W_k}\right) +$$

$$+ \frac{\partial A_i}{\partial L_{2i}}\frac{\partial B_i}{\partial L_{2i}}\left(\frac{\partial A_i}{\partial P_i}\frac{\partial B_i}{\partial W_k} + (n-1)\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial W_k}\right) - \frac{\partial A_i}{\partial L_{2i}}\frac{\partial A_i}{\partial L_{2i}}\left(\frac{\partial B_i}{\partial P_i}\frac{\partial B_i}{\partial W_k} + (n-1)\frac{\partial B_i}{\partial P_k}\frac{\partial B_i}{\partial W_k}\right) -$$

$$\left. -\frac{\partial B_i}{\partial L_{2i}}\frac{\partial B_i}{\partial L_{2i}}\left(\frac{\partial A_i}{\partial P_i}\frac{\partial A_i}{\partial W_k} + (n-1)\frac{\partial A_i}{\partial P_k}\frac{\partial A_i}{\partial W_k}\right) + \frac{\partial B_i}{\partial L_{2i}}\frac{\partial A_i}{\partial L_{2i}}\left(\frac{\partial B_i}{\partial P_i}\frac{\partial A_i}{\partial W_k} + (n-1)\frac{\partial B_i}{\partial P_k}\frac{\partial A_i}{\partial W_k}\right)\right].$$

Eliminating again the repeated terms:

$$\frac{dP_k}{dW_i} = \frac{1}{\frac{\partial B_i}{\partial L_{2_i}}\left(\frac{\partial A_i}{\partial P_i} + (n-1)\frac{\partial A_i}{\partial P_k}\right) - \frac{\partial A_i}{\partial L_{2_i}}\left(\frac{\partial B_i}{\partial P_i} + (n-1)\frac{\partial B_i}{\partial P_k}\right)} \times$$

$$\times \frac{1}{\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k} - \frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}} - \frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_i} + \frac{\partial A_i}{\partial P_i}\frac{\partial B_i}{\partial L_{2_i}}} \times$$

$$\times \left[\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial L_{2_i}}\left(\frac{\partial A_i}{\partial P_i}\frac{\partial B_i}{\partial W_k} - \frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial W_i}\right) - \frac{\partial A_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial L_{2_i}}\left(\frac{\partial B_i}{\partial P_i}\frac{\partial B_i}{\partial W_k} - \frac{\partial B_i}{\partial P_k}\frac{\partial B_i}{\partial W_i}\right) - \right.$$

$$\left. - \frac{\partial B_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial L_{2_i}}\left(\frac{\partial A_i}{\partial P_i}\frac{\partial A_i}{\partial W_k} - \frac{\partial A_i}{\partial P_k}\frac{\partial A_i}{\partial W_i}\right) + \frac{\partial B_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial L_{2_i}}\left(\frac{\partial B_i}{\partial P_i}\frac{\partial A_i}{\partial W_k} - \frac{\partial B_i}{\partial P_k}\frac{\partial A_i}{\partial W_i}\right)\right]. \quad (C.2)$$

If once again we use the information provided in Table 9 (Appendix D), we verify that the sign of dP_k/dW_i is undetermined, as it depends on the relative dimension of the partial derivatives of functions A_i and B_i with respect to prices and wages. Nevertheless we can still analyze the expressions of the derivatives in Table 1 to identify the characteristics of the industry that are more likely to affect the sign of dP_k/dW_i.

We conclude that for any $k \neq i$ we have:

$$\begin{cases} \frac{\partial P_k(.)}{\partial W_i} < 0, & \textit{if industry} = \textit{type A} \\ \frac{\partial P_k(.)}{\partial W_i} > 0, & \textit{if industry} = \textit{type B} \end{cases},$$

where in *type A* industries $\frac{\partial g_i(.)}{\partial W_i}$ and $\frac{\partial f_i(.)}{\partial P_k}$ are sufficiently large relatively to $\frac{\partial g_i(.)}{\partial W_k}$ and $\frac{\partial f_i(.)}{\partial P_i}$, while in *type B* industries the opposite occurs.

Appendix D

Derivative	Value	Sign
$\frac{\partial A_i}{\partial P_i}$	$2\frac{\partial f_i(.)}{\partial P_i} + \left(P_i - \bar{W}\frac{\partial L_{2i}}{\partial h_i}\right)\frac{\partial^2 f_i(.)}{\partial {P_i}^2}$	≤ 0
$\frac{\partial A_i}{\partial P_k}$	$\frac{\partial f_i(.)}{\partial P_k} + \left(P_i - \bar{W}\frac{\partial L_{2i}}{\partial h_i}\right)\frac{\partial^2 f_i(.)}{\partial P_i \partial P_k}$	≥ 0
$\frac{\partial A_i}{\partial P_i} + (n-1)\frac{\partial A_i}{\partial P_k}$	$\frac{\partial f_i(.)}{\partial P_i} + \frac{\partial F(.)}{\partial P_i} + \left(P_i - \bar{W}\frac{\partial L_{2i}}{\partial h_i}\right)\left[\frac{\partial^2 f_i(.)}{\partial {P_i}^2} + (n-1)\frac{\partial^2 f_i(.)}{\partial P_i \partial P_k}\right]$	≤ 0
$\frac{\partial A_i}{\partial L_{2i}}$	$\bar{W}\frac{\partial f_i(.)}{\partial P_i}\frac{\partial^2 h_i(.)}{\partial (L_{2i})^2}\left(\frac{\partial L_{2i}}{\partial h_i}\right)^2$	≥ 0
$\frac{\partial A_i}{\partial W_i}$	$\bar{W}\frac{\partial f_i(.)}{\partial P_i}\frac{\partial^2 h_i(.)}{\partial L_{2i}\partial g_i}\frac{\partial g_i(.)}{\partial W_i}\left(\frac{\partial L_{2i}}{\partial h_i}\right)^2$	≤ 0
$\frac{\partial A_i}{\partial W_k}$	$\bar{W}\frac{\partial f_i(.)}{\partial P_i}\frac{\partial^2 h_i(.)}{\partial L_{2i}\partial g_i}\frac{\partial g_i(.)}{\partial W_k}\left(\frac{\partial L_{2i}}{\partial h_i}\right)^2$	≥ 0
$\frac{\partial A_i}{\partial W_i} + (n-1)\frac{\partial A_i}{\partial W_k}$	$\bar{W}\frac{\partial f_i(.)}{\partial P_i}\frac{\partial^2 h_i(.)}{\partial L_{2i}\partial g_i}\frac{\partial G(.)}{\partial W_i}\left(\frac{\partial L_{2i}}{\partial h_i}\right)^2$	≤ 0
$\frac{\partial B_i}{\partial P_i}$	$\frac{\partial f_i(.)}{\partial P_i}$	≤ 0
$\frac{\partial B_i}{\partial P_k}$	$\frac{\partial f_i(.)}{\partial P_k}$	≥ 0
$\frac{\partial B_i}{\partial P_i} + (n-1)\frac{\partial B_i}{\partial P_k}$	$\frac{\partial F(.)}{\partial P_i}$	≤ 0
$\frac{\partial B_i}{\partial L_{2i}}$	$-\frac{\partial h_i(.)}{\partial L_{2i}}$	≤ 0
$\frac{\partial B_i}{\partial W_i}$	$-\frac{\partial h_i(.)}{\partial g_i}\frac{\partial g_i(.)}{\partial W_i}$	≤ 0
$\frac{\partial B_i}{\partial W_k}$	$-\frac{\partial h_i(.)}{\partial g_i}\frac{\partial g_i(.)}{\partial W_k}$	≥ 0
$\frac{\partial B_i}{\partial W_i} + (n-1)\frac{\partial B_i}{\partial W_k}$	$-\frac{\partial h_i(.)}{\partial g_i}\frac{\partial G(.)}{\partial W_i}$	≤ 0

Table 9 – Partial derivatives

Appendix E

Here we prove that when firms are symmetric the term $\sum_{k\neq j}^{n-1}\frac{\partial f_j(.)}{\partial P_k}\frac{\partial P_k(.)}{\partial W_i}$ is always negative:

$$\sum_{k\neq j}^{n-1}\frac{\partial f_j(.)}{\partial P_k}\frac{\partial P_k(.)}{\partial W_i}=\frac{\partial f_j(.)}{\partial P_k}\left(\frac{\partial P_i(.)}{\partial W_i}+(n-2)\frac{\partial P_k(.)}{\partial W_i}\right)=$$

$$=\frac{\partial f_j(.)}{\partial P_i}\left(\frac{\partial P_i(.)}{\partial W_i}+(n-2)\frac{\partial P_i(.)}{\partial W_k}\right)$$

Replacing $\partial P_i(.)/\partial W_i$ and $\partial P_k(.)/\partial W_i$ respectively by equations (C.3) and (C.4) in Appendix C:

$$\frac{\partial f_j(.)}{\partial P_i}\times\frac{1}{\frac{\partial B_i}{\partial L_{2_i}}\left(\frac{\partial A_i}{\partial P_i}+(n-1)\frac{\partial A_i}{\partial P_k}\right)-\frac{\partial A_i}{\partial L_{2_i}}\left(\frac{\partial B_i}{\partial P_i}+(n-1)\frac{\partial B_i}{\partial P_k}\right)}\times$$

$$\times\frac{1}{\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k}-\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}}-\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_i}+\frac{\partial A_i}{\partial P_i}\frac{\partial B_i}{\partial L_{2_i}}}\times$$

$$\times\left[-\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}}\left(\frac{\partial B_i}{\partial W_i}+(n-1)\frac{\partial B_i}{\partial W_k}\right)+\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k}\left(\frac{\partial B_i}{\partial W_i}+(n-1)\frac{\partial B_i}{\partial W_k}\right)+\right.$$

$$+\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}}\left(\frac{\partial A_i}{\partial W_i}+(n-1)\frac{\partial A_i}{\partial W_k}\right)-\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k}\left(\frac{\partial A_i}{\partial W_i}+(n-1)\frac{\partial A_i}{\partial W_k}\right)+$$

$$+\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial W_i}\frac{\partial B_i}{\partial L_{2_i}}\left(\frac{\partial A_i}{\partial P_i}+(n-1)\frac{\partial A_i}{\partial P_k}\right)-\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial W_i}\frac{\partial A_i}{\partial L_{2_i}}\left(\frac{\partial B_i}{\partial P_i}+(n-1)\frac{\partial B_i}{\partial P_k}\right)-$$

$$-\frac{\partial A_i}{\partial W_i}\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial L_{2_i}}\left(\frac{\partial A_i}{\partial P_i}+(n-1)\frac{\partial A_i}{\partial P_k}\right)+\frac{\partial A_i}{\partial W_i}\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial L_{2_i}}\left(\frac{\partial B_i}{\partial P_i}+(n-1)\frac{\partial B_i}{\partial P_k}\right)+$$

$$+(n-2)\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial L_{2_i}}\left(\frac{\partial A_i}{\partial P_i}\frac{\partial B_i}{\partial W_k}-\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial W_i}\right)-(n-2)\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial L_{2_i}}\left(\frac{\partial B_i}{\partial P_i}\frac{\partial B_i}{\partial W_k}-\frac{\partial B_i}{\partial P_k}\frac{\partial B_i}{\partial W_i}\right)-$$

$$\left.-(n-2)\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial L_{2_i}}\left(\frac{\partial A_i}{\partial P_i}\frac{\partial A_i}{\partial W_k}-\frac{\partial A_i}{\partial P_k}\frac{\partial A_i}{\partial W_i}\right)+(n-2)\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial L_{2_i}}\left(\frac{\partial B_i}{\partial P_i}\frac{\partial A_i}{\partial W_k}-\frac{\partial B_i}{\partial P_k}\frac{\partial A_i}{\partial W_i}\right)\right]=$$

$$= \frac{\partial f_j(.)}{\partial P_i} \times \frac{1}{\frac{\partial B_i}{\partial L_{2_i}}\left(\frac{\partial A_i}{\partial P_i} + (n-1)\frac{\partial A_i}{\partial P_k}\right) - \frac{\partial A_i}{\partial L_{2_i}}\left(\frac{\partial B_i}{\partial P_i} + (n-1)\frac{\partial B_i}{\partial P_k}\right)} \times$$

$$\times \frac{1}{\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k} - \frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}} - \frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_i} + \frac{\partial A_i}{\partial P_i}\frac{\partial B_i}{\partial L_{2_i}}} \times$$

$$\times \left[-\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial W_k}\left(\frac{\partial A_i}{\partial P_i} + (n-1)\frac{\partial A_i}{\partial P_k}\right) + \frac{\partial A_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial W_k}\left(\frac{\partial B_i}{\partial P_i} + (n-1)\frac{\partial B_i}{\partial P_k}\right) + \right.$$

$$+ \frac{\partial B_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial W_k}\left(\frac{\partial A_i}{\partial P_i} + (n-1)\frac{\partial A_i}{\partial P_k}\right) - \frac{\partial B_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial W_k}\left(\frac{\partial B_i}{\partial P_i} + (n-1)\frac{\partial B_i}{\partial P_k}\right) +$$

$$+ \frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial P_i}\left(\frac{\partial B_i}{\partial W_i} + (n-1)\frac{\partial B_i}{\partial W_k}\right) - \frac{\partial A_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_i}\left(\frac{\partial B_i}{\partial W_i} + (n-1)\frac{\partial B_i}{\partial W_k}\right) -$$

$$\left. - \frac{\partial B_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial P_i}\left(\frac{\partial A_i}{\partial W_i} + (n-1)\frac{\partial A_i}{\partial W_k}\right) + \frac{\partial B_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_i}\left(\frac{\partial A_i}{\partial W_i} + (n-1)\frac{\partial A_i}{\partial W_k}\right)\right].$$

Given the signs of the partial derivatives of functions A and B provided in Table 9 in Appendix D, we conclude that:

$$\sum_{k \neq j}^{n-1} \frac{\partial f_j(.)}{\partial P_k}\frac{\partial P_k(.)}{\partial W_i} < 0.$$

Appendix F

In this appendix we show that $\frac{\partial L}{\partial W_i} - \frac{\partial L_i}{\partial W_i}$ is negative in *type B* industries (industries where $\partial P_k(.)/\partial W_i > 0$).

$$\frac{\partial L}{\partial W_i} - \frac{\partial L_i}{\partial W_i} = \sum_{j \neq i}^{n-1} \left\{ \left(P_j(.) - \bar{W}\frac{\partial L_{2j}(.)}{\partial h_j} \right) \sum_{k \neq j}^{n-1} \frac{\partial f_j(.)}{\partial P_k} \frac{\partial P_k(.)}{\partial W_i} \right\} +$$

$$+ \sum_{j \neq i}^{n-1} \left\{ \left(\bar{W}\frac{\partial L_{2j}(.)}{\partial h_j} \frac{\partial h_j(.)}{\partial g_j} - W_j \right) \frac{\partial g_j(.)}{\partial W_i} \right\}.$$

Under symmetry:

$$\frac{\partial L}{\partial W_i} - \frac{\partial L_i}{\partial W_i} = (n-1)\left(P_j(.) - \bar{W}\frac{\partial L_{2j}(.)}{\partial h_j} \right) \sum_{k \neq j}^{n-1} \frac{\partial f_j(.)}{\partial P_k} \frac{\partial P_k(.)}{\partial W_i} +$$

$$+ \left(\bar{W}\frac{\partial L_{2j}(.)}{\partial h_j} \frac{\partial h_j(.)}{\partial g_j} - W_j \right)(n-1)\frac{\partial g_j(.)}{\partial W_i}.$$

Using the fact that under symmetry $\frac{\partial G(.)}{\partial W_i} = \frac{\partial g_i(.)}{\partial W_i} + (n-1)\frac{\partial g_j(.)}{\partial W_i}$:

$$\frac{\partial L}{\partial W_i} - \frac{\partial L_i}{\partial W_i} = (n-1)\left(P_j(.) - \bar{W}\frac{\partial L_{2j}(.)}{\partial h_j} \right) \sum_{k \neq j}^{n-1} \frac{\partial f_j(.)}{\partial P_k} \frac{\partial P_k(.)}{\partial W_i} +$$

$$+ \left(\bar{W}\frac{\partial L_{2j}(.)}{\partial h_j} \frac{\partial h_j(.)}{\partial g_j} - W_j \right)\left(\frac{\partial G(.)}{\partial W_i} - \frac{\partial g_i(.)}{\partial W_i} \right).$$

Equation (3.4.1) can now be used to replace the term $\left(\bar{W}\frac{\partial L_{2j}(.)}{\partial h_j} \frac{\partial h_j(.)}{\partial g_j} - W_j \right)\frac{\partial g_i(.)}{\partial W_i}$:

$$\frac{\partial L}{\partial W_i} - \frac{\partial L_i}{\partial W_i} = (n-1)\left(P_j(.) - \bar{W}\frac{\partial L_{2j}(.)}{\partial h_j} \right) \sum_{k \neq j}^{n-1} \frac{\partial f_j(.)}{\partial P_k} \frac{\partial P_k(.)}{\partial W_i} +$$

$$+\left(\bar{W}\frac{\partial L_{2j}(.)}{\partial h_j}\frac{\partial h_j(.)}{\partial g_j}-W_j\right)\frac{\partial G(.)}{\partial W_i}+\left(P_i(.)-\bar{W}\frac{\partial L_{2i}(.)}{\partial h_i}\right)\sum_{k\neq i}^{n-1}\left\{\frac{\partial f_i(.)}{\partial P_k}\frac{\partial P_k(.)}{\partial W_i}\right\}$$
$$-g_i(.).$$

Dividing out the common factor from the first and third terms:

$$\frac{\partial L}{\partial W_i}-\frac{\partial L_i}{\partial W_i}=\left(P_j(.)-\bar{W}\frac{\partial L_{2j}(.)}{\partial h_j}\right)\left[\sum_{k\neq i}^{n-1}\left\{\frac{\partial f_i(.)}{\partial P_k}\frac{\partial P_k(.)}{\partial W_i}\right\}+(n-1)\sum_{k\neq j}^{n-1}\frac{\partial f_j(.)}{\partial P_k}\frac{\partial P_k(.)}{\partial W_i}\right]+$$
$$+\left(\bar{W}\frac{\partial L_{2j}(.)}{\partial h_j}\frac{\partial h_j(.)}{\partial g_j}-W_j\right)\frac{\partial G(.)}{\partial W_i}-g_j(.).$$

Applying symmetry and factoring out the common factor from the terms in brackets:

$$\frac{\partial L}{\partial W_i}-\frac{\partial L_i}{\partial W_i}=$$
$$=\left(P_j(.)-\bar{W}\frac{\partial L_{2j}(.)}{\partial h_j}\right)\frac{\partial f_j(.)}{\partial P_k}\left[(n-1)\frac{\partial P_k(.)}{\partial W_i}+(n-1)\left(\frac{\partial P_i(.)}{\partial W_i}+(n-2)\frac{\partial P_k(.)}{\partial W_i}\right)\right]+$$
$$+\left(\bar{W}\frac{\partial L_{2j}(.)}{\partial h_j}\frac{\partial h_j(.)}{\partial g_j}-W_j\right)\frac{\partial G(.)}{\partial W_i}-g_j(.)=$$

$$=\left(P_j(.)-\bar{W}\frac{\partial L_{2j}(.)}{\partial h_j}\right)\frac{\partial f_j(.)}{\partial P_k}(n-1)\left(\frac{\partial P_i(.)}{\partial W_i}+(n-1)\frac{\partial P_k(.)}{\partial W_i}\right)+$$
$$+\left(\bar{W}\frac{\partial L_{2j}(.)}{\partial h_j}\frac{\partial h_j(.)}{\partial g_j}-W_j\right)\frac{\partial G(.)}{\partial W_i}-g_j(.).$$

The last expression is negative as long as $\frac{\partial P_i(.)}{\partial W_i}+(n-1)\frac{\partial P_k(.)}{\partial W_i}<0$, which we also prove here using equations (C.3) and (C.4) from Appendix C:

$$\frac{\partial P_i(.)}{\partial W_i}+(n-1)\frac{\partial P_k(.)}{\partial W_i}=$$

$$=\frac{1}{\frac{\partial B_i}{\partial L_{2_i}}\left(\frac{\partial A_i}{\partial P_i}+(n-1)\frac{\partial A_i}{\partial P_k}\right)-\frac{\partial A_i}{\partial L_{2_i}}\left(\frac{\partial B_i}{\partial P_i}+(n-1)\frac{\partial B_i}{\partial P_k}\right)}\times$$

$$\times\frac{1}{\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k}-\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}}-\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_i}+\frac{\partial A_i}{\partial P_i}\frac{\partial B_i}{\partial L_{2_i}}}\times$$

$$\times\left[-\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}}\left(\frac{\partial B_i}{\partial W_i}+(n-1)\frac{\partial B_i}{\partial W_k}\right)+\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k}\left(\frac{\partial B_i}{\partial W_i}+(n-1)\frac{\partial B_i}{\partial W_k}\right)+\right.$$

$$+\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}}\left(\frac{\partial A_i}{\partial W_i}+(n-1)\frac{\partial A_i}{\partial W_k}\right)-\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k}\left(\frac{\partial A_i}{\partial W_i}+(n-1)\frac{\partial A_i}{\partial W_k}\right)+$$

$$+\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial W_i}\frac{\partial B_i}{\partial L_{2_i}}\left(\frac{\partial A_i}{\partial P_i}+(n-1)\frac{\partial A_i}{\partial P_k}\right)-\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial W_i}\frac{\partial A_i}{\partial L_{2_i}}\left(\frac{\partial B_i}{\partial P_i}+(n-1)\frac{\partial B_i}{\partial P_k}\right)-$$

$$-\frac{\partial A_i}{\partial W_i}\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial L_{2_i}}\left(\frac{\partial A_i}{\partial P_i}+(n-1)\frac{\partial A_i}{\partial P_k}\right)+\frac{\partial A_i}{\partial W_i}\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial L_{2_i}}\left(\frac{\partial B_i}{\partial P_i}+(n-1)\frac{\partial B_i}{\partial P_k}\right)+$$

$$+(n-1)\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial L_{2_i}}\left(\frac{\partial A_i}{\partial P_i}\frac{\partial B_i}{\partial W_k}-\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial W_i}\right)-(n-1)\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial L_{2_i}}\left(\frac{\partial B_i}{\partial P_i}\frac{\partial B_i}{\partial W_k}-\frac{\partial B_i}{\partial P_k}\frac{\partial B_i}{\partial W_i}\right)-$$

$$\left.-(n-1)\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial L_{2_i}}\left(\frac{\partial A_i}{\partial P_i}\frac{\partial A_i}{\partial W_k}-\frac{\partial A_i}{\partial P_k}\frac{\partial A_i}{\partial W_i}\right)+(n-1)\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial L_{2_i}}\left(\frac{\partial B_i}{\partial P_i}\frac{\partial A_i}{\partial W_k}-\frac{\partial B_i}{\partial P_k}\frac{\partial A_i}{\partial W_i}\right)\right].$$

Eliminating the repeated terms:

$$\frac{\partial P_i(.)}{\partial W_i}+(n-1)\frac{\partial P_k(.)}{\partial W_i}=$$

$$=\frac{1}{\frac{\partial B_i}{\partial L_{2_i}}\left(\frac{\partial A_i}{\partial P_i}+(n-1)\frac{\partial A_i}{\partial P_k}\right)-\frac{\partial A_i}{\partial L_{2_i}}\left(\frac{\partial B_i}{\partial P_i}+(n-1)\frac{\partial B_i}{\partial P_k}\right)}\times$$

$$\times\frac{1}{\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k}-\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}}-\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_i}+\frac{\partial A_i}{\partial P_i}\frac{\partial B_i}{\partial L_{2_i}}}\times$$

$$\times\left[-\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}}\left(\frac{\partial B_i}{\partial W_i}+(n-1)\frac{\partial B_i}{\partial W_k}\right)+\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k}\left(\frac{\partial B_i}{\partial W_i}+(n-1)\frac{\partial B_i}{\partial W_k}\right)+\right.$$

$$+\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}}\left(\frac{\partial A_i}{\partial W_i}+(n-1)\frac{\partial A_i}{\partial W_k}\right)-\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k}\left(\frac{\partial A_i}{\partial W_i}+(n-1)\frac{\partial A_i}{\partial W_k}\right)+$$

$$+\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial P_i}\left(\frac{\partial B_i}{\partial W_i}+(n-1)\frac{\partial B_i}{\partial W_k}\right)-\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_i}\left(\frac{\partial B_i}{\partial W_i}+(n-1)\frac{\partial B_i}{\partial W_k}\right)-$$

$$-\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial P_i}\left(\frac{\partial A_i}{\partial W_i}+(n-1)\frac{\partial A_i}{\partial W_k}\right)+\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_i}\left(\frac{\partial A_i}{\partial W_i}+(n-1)\frac{\partial A_i}{\partial W_k}\right)\Bigg].$$

Given the information in Table 9 in Appendix D, the last expression for $\frac{\partial P_i(.)}{\partial W_i}+(n-1)\frac{\partial P_k(.)}{\partial W_i}$ is negative and hence $\frac{\partial L}{\partial W_i}-\frac{\partial L_i}{\partial W_i}$ is negative in *type B* industries.

Appendix G

We prove now that when all the firms of the industry increase the wage in the same amount at the first stage of the game, the best reply for any firm is to decrease the price at the second stage. Under symmetry,

$$\sum_{k=1}^{n}\frac{\partial P_i(.)}{\partial W_k}=\frac{\partial P_i(.)}{\partial W_i}+(n-1)\frac{\partial P_i(.)}{\partial W_k}=$$

$$=\frac{\partial P_i(.)}{\partial W_i}+(n-1)\frac{\partial P_k(.)}{\partial W_i}.$$

Replacing $\frac{\partial P_i(.)}{\partial W_i}$ and $\frac{\partial P_k(.)}{\partial W_i}$ by equations (C.3) and (C.4) respectively:

$$\sum_{k=1}^{n}\frac{\partial P_i(.)}{\partial W_k}=\frac{1}{\frac{\partial B_i}{\partial L_{2_i}}\left(\frac{\partial A_i}{\partial P_i}+(n-1)\frac{\partial A_i}{\partial P_k}\right)-\frac{\partial A_i}{\partial L_{2_i}}\left(\frac{\partial B_i}{\partial P_i}+(n-1)\frac{\partial B_i}{\partial P_k}\right)}\times$$

$$\times\frac{1}{\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k}-\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}}-\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_i}+\frac{\partial A_i}{\partial P_i}\frac{\partial B_i}{\partial L_{2_i}}}\times$$

$$\times\left[-\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}}\left(\frac{\partial B_i}{\partial W_i}+(n-1)\frac{\partial B_i}{\partial W_k}\right)+\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k}\left(\frac{\partial B_i}{\partial W_i}+(n-1)\frac{\partial B_i}{\partial W_k}\right)+\right.$$

$$+\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}}\left(\frac{\partial A_i}{\partial W_i}+(n-1)\frac{\partial A_i}{\partial W_k}\right)-\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k}\left(\frac{\partial A_i}{\partial W_i}+(n-1)\frac{\partial A_i}{\partial W_k}\right)+$$

$$+\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial W_i}\frac{\partial B_i}{\partial L_{2_i}}\left(\frac{\partial A_i}{\partial P_i}+(n-1)\frac{\partial A_i}{\partial P_k}\right)-\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial W_i}\frac{\partial A_i}{\partial L_{2_i}}\left(\frac{\partial B_i}{\partial P_i}+(n-1)\frac{\partial B_i}{\partial P_k}\right)-$$

$$-\frac{\partial A_i}{\partial W_i}\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial L_{2_i}}\left(\frac{\partial A_i}{\partial P_i}+(n-1)\frac{\partial A_i}{\partial P_k}\right)+\frac{\partial A_i}{\partial W_i}\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial L_{2_i}}\left(\frac{\partial B_i}{\partial P_i}+(n-1)\frac{\partial B_i}{\partial P_k}\right)+$$

$$+(n-1)\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial L_{2_i}}\left(\frac{\partial A_i}{\partial P_i}\frac{\partial B_i}{\partial W_k}-\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial W_i}\right)-(n-1)\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial L_{2_i}}\left(\frac{\partial B_i}{\partial P_i}\frac{\partial B_i}{\partial W_k}-\frac{\partial B_i}{\partial P_k}\frac{\partial B_i}{\partial W_i}\right)-$$

$$\left.-(n-1)\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial L_{2_i}}\left(\frac{\partial A_i}{\partial P_i}\frac{\partial A_i}{\partial W_k}-\frac{\partial A_i}{\partial P_k}\frac{\partial A_i}{\partial W_i}\right)+(n-1)\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial L_{2_i}}\left(\frac{\partial B_i}{\partial P_i}\frac{\partial A_i}{\partial W_k}-\frac{\partial B_i}{\partial P_k}\frac{\partial A_i}{\partial W_i}\right)\right].$$

The last equation can be simplified to:

$$\sum_{k=1}^{n}\frac{\partial P_i(.)}{\partial W_k}=\frac{1}{\frac{\partial B_i}{\partial L_{2_i}}\left(\frac{\partial A_i}{\partial P_i}+(n-1)\frac{\partial A_i}{\partial P_k}\right)-\frac{\partial A_i}{\partial L_{2_i}}\left(\frac{\partial B_i}{\partial P_i}+(n-1)\frac{\partial B_i}{\partial P_k}\right)}\times$$

$$\times\frac{1}{\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k}-\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}}-\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_i}+\frac{\partial A_i}{\partial P_i}\frac{\partial B_i}{\partial L_{2_i}}}\times$$

$$\times\left[-\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}}\left(\frac{\partial B_i}{\partial W_i}+(n-1)\frac{\partial B_i}{\partial W_k}\right)+\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k}\left(\frac{\partial B_i}{\partial W_i}+(n-1)\frac{\partial B_i}{\partial W_k}\right)+\right.$$

$$+\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial P_k}\frac{\partial B_i}{\partial L_{2_i}}\left(\frac{\partial A_i}{\partial W_i}+(n-1)\frac{\partial A_i}{\partial W_k}\right)-\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_k}\left(\frac{\partial A_i}{\partial W_i}+(n-1)\frac{\partial A_i}{\partial W_k}\right)+$$

$$+\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial P_i}\left(\frac{\partial B_i}{\partial W_i}+(n-1)\frac{\partial B_i}{\partial W_k}\right)-\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_i}\left(\frac{\partial B_i}{\partial W_i}+(n-1)\frac{\partial B_i}{\partial W_k}\right)-$$

$$\left.-\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial P_i}\left(\frac{\partial A_i}{\partial W_i}+(n-1)\frac{\partial A_i}{\partial W_k}\right)+\frac{\partial B_i}{\partial L_{2_i}}\frac{\partial A_i}{\partial L_{2_i}}\frac{\partial B_i}{\partial P_i}\left(\frac{\partial A_i}{\partial W_i}+(n-1)\frac{\partial A_i}{\partial W_k}\right)\right].$$

Given the sign of the derivatives in Table 9 in Appendix D, the last expression is negative and the assertion is complete.

Appendix H

We provide in this appendix a formal demonstration that in *type B* industries the wages centrally determined under collusion are lower than the wages fixed at the simultaneous non-cooperative equilibrium.

$$\frac{\partial L}{\partial W_i} - \frac{\partial L_i}{\partial W_i} = \sum_{j \neq i}^{n-1} \left\{ \left(P_j(.) - \overline{W} \frac{\partial L_{2j}(.)}{\partial h_j} \right) \sum_{k \neq j}^{n-1} \frac{\partial f_j(.)}{\partial P_k} \frac{\partial P_k(.)}{\partial W_i} \right\} +$$

$$+ \left(P_i(.) - \overline{W} \frac{\partial L_{2i}(.)}{\partial h_i} \right) \sum_{k \neq i}^{n-1} \frac{\partial f_i(.)}{\partial P_k} \frac{\partial P_k(.)}{\partial W_i} + \sum_{j \neq i}^{n-1} \left\{ \left(\overline{W} \frac{\partial L_{2j}(.)}{\partial h_j} \frac{\partial h_j(.)}{\partial g_j} - W_j \right) \frac{\partial g_j(.)}{\partial W_i} \right\}.$$

Under symmetry,

$$\frac{\partial L}{\partial W_i} - \frac{\partial L_i}{\partial W_i} = (n-1) \left(P_j(.) - \overline{W} \frac{\partial L_{2j}(.)}{\partial h_j} \right) \sum_{k \neq j}^{n-1} \frac{\partial f_j(.)}{\partial P_k} \frac{\partial P_k(.)}{\partial W_i} +$$

$$+ \left(P_j(.) - \overline{W} \frac{\partial L_{2j}(.)}{\partial h_j} \right) \sum_{k \neq i}^{n-1} \frac{\partial f_i(.)}{\partial P_k} \frac{\partial P_k(.)}{\partial W_i} + \left(\overline{W} \frac{\partial L_{2j}(.)}{\partial h_j} \frac{\partial h_j(.)}{\partial g_j} - W_j \right) (n-1) \frac{\partial g_j(.)}{\partial W_i}.$$

Factoring out the common factor from the first and second terms and using the fact that $\frac{\partial G(.)}{\partial W_i} = \frac{\partial g_i(.)}{\partial W_i} + (n-1) \frac{\partial g_j(.)}{\partial W_i}$:

$$\frac{\partial L}{\partial W_i} - \frac{\partial L_i}{\partial W_i} = \left(P_j(.) - \overline{W} \frac{\partial L_{2j}(.)}{\partial h_j} \right) \left[\sum_{k \neq i}^{n-1} \frac{\partial f_i(.)}{\partial P_k} \frac{\partial P_k(.)}{\partial W_i} + (n-1) \sum_{k \neq j}^{n-1} \frac{\partial f_j(.)}{\partial P_k} \frac{\partial P_k(.)}{\partial W_i} \right] +$$

$$+ \left(\overline{W} \frac{\partial L_{2j}(.)}{\partial h_j} \frac{\partial h_j(.)}{\partial g_j} - W_j \right) \left(\frac{\partial G(.)}{\partial W_i} - \frac{\partial g_i(.)}{\partial W_i} \right).$$

Substituting equation (3.7.1) in the last expression:

$$\frac{\partial L}{\partial W_i} - \frac{\partial L_i}{\partial W_i} = \left(P_j(.) - \bar{W}\frac{\partial L_{2j}(.)}{\partial h_j}\right)\left[\sum_{k\neq i}^{n-1}\frac{\partial f_i(.)}{\partial P_k}\frac{\partial P_k(.)}{\partial W_i} + (n-1)\sum_{k\neq j}^{n-1}\frac{\partial f_j(.)}{\partial P_k}\frac{\partial P_k(.)}{\partial W_i}\right] +$$

$$+\left(\bar{W}\frac{\partial L_{2j}(.)}{\partial h_j}\frac{\partial h_j(.)}{\partial g_j} - W_j\right)\frac{\partial G(.)}{\partial W_i} - g_j(.).$$

Applying symmetry to the terms in brackets:

$$\frac{\partial L}{\partial W_i} - \frac{\partial L_i}{\partial W_i} = \left(P_j(.) - \bar{W}\frac{\partial L_{2j}(.)}{\partial h_j}\right)\frac{\partial f_j(.)}{\partial P_k}\left[(n-1)\frac{\partial P_k(.)}{\partial W_i} + (n-1)\left(\frac{\partial P_i(.)}{\partial W_i} + (n-2)\frac{\partial P_k(.)}{\partial W_i}\right)\right]$$

$$+\left(\bar{W}\frac{\partial L_{2j}(.)}{\partial h_j}\frac{\partial h_j(.)}{\partial g_j} - W_j\right)\frac{\partial G(.)}{\partial W_i} - g_j(.) \quad \Leftrightarrow$$

$$\Leftrightarrow \quad \frac{\partial L}{\partial W_i} - \frac{\partial L_i}{\partial W_i} = \left(P_j(.) - \bar{W}\frac{\partial L_{2j}(.)}{\partial h_j}\right)\frac{\partial f_j(.)}{\partial P_i}(n-1)\left[\frac{\partial P_i(.)}{\partial W_i} + (n-1)\frac{\partial P_j(.)}{\partial W_i}\right] +$$

$$+\left(\bar{W}\frac{\partial L_{2j}(.)}{\partial h_j}\frac{\partial h_j(.)}{\partial g_j} - W_j\right)\frac{\partial G(.)}{\partial W_i} - g_j(.).$$

Because $\frac{\partial P_i(.)}{\partial W_i} + (n-1)\frac{\partial P_j(.)}{\partial W_i}$ is always negative (see Appendix F) and $\bar{W}\frac{\partial L_{2j}(.)}{\partial h_j}\frac{\partial h_j(.)}{\partial g_j} - W_j$ is negative in *type B* industries, the differential $\frac{\partial L}{\partial W_i} - \frac{\partial L_i}{\partial W_i}$ is also negative. It is hence proved that the wages fixed at the collusive equilibrium are lower than the ones determined at the simultaneous non-cooperative equilibrium.

Appendix I

Matlab code for the traditional OLS EM algorithm

```
function output = em0(DEP,INDS,IND)

% Traditional OLS EM Algorithm
%---------------------------------------------------
% output = em0(DEP,INDS,IND,S)
% DEP = Dependent variable vector
% INDS = Switching independent variables vector
% IND = Non-switching independent variables vector
%---------------------------------------------------

% Example
%---------------------------------------------------
% Supply equation: P = CMG + c(1)*S*Q + C(2)*(1-S)*Q + U2,
%         where S is the unobserved state variable
%
% Command to estimate the supply equation as a switching regression:
% output = em1(P,Q,CMG)
%---------------------------------------------------

% Maximum number of iterations
maxit = 100;

% Dimensions
T = size(DEP,1);
k = 2*size(INDS,2)+size(IND,2);

% Initial Values
%---------------------------------------------------
% Note: OLS function is available in Appendix L
ols=feval('ols',DEP,[INDS IND]);
if mean(DEP-[INDS IND]*ols.beta)<median(DEP-[INDS IND]*ols.beta);
    crit0 = 1-normcdf(ones(T,1)*prctile(DEP-[INDS IND]*...
        ols.beta,25),DEP-[INDS IND]*ols.beta,sqrt(ols.sige));
```

```
else
    crit0 = 1-normcdf(ones(T,1)*prctile(DEP-[INDS IND]*...
        ols.beta,75),DEP-[INDS IND]*ols.beta,sqrt(ols.sige));
end
%---------------------------------------------------

% Creation of variables
%---------------------------------------------------
W = zeros(T,maxit);
Shat = zeros(T,maxit);
lambda = zeros(1,maxit);
beta = zeros(k,maxit);
sigma = zeros(1,maxit);
r2 = zeros(1,maxit);
%---------------------------------------------------

% First iteration
%---------------------------------------------------
iter = 1;
% Expectation step
W(:,1) = crit0;
Shat(W(:,1)>0.5,1) = 1;
lambda(1) = sum(W(:,1))/T;
% Maximization step
ols=feval('ols',DEP,[INDS.*W(:,1) INDS.*(1-W(:,1)) IND]);
beta(:,1) = ols.beta;
sigma(1) = ols.sige;
r2(1) = ols.rsqr;
%---------------------------------------------------

% Second and remaining iterations
%---------------------------------------------------
crit=0;
while  crit < 0.99999;
    iter = iter+1;
    % Expectation step
    h0 = normpdf(DEP,[zeros(T,1) INDS IND]*beta(:,iter-1),...
        sqrt(sigma(1,iter-1)));
    h1 = normpdf(DEP,[INDS zeros(T,1) IND]*beta(:,iter-1),...
```

```
        sqrt(sigma(1,iter-1)));
    W(:,iter) = lambda(iter-1)*h1 ./ ( lambda(iter-1)*h1 +...
        (1-lambda(iter-1))*h0 );
    Shat(W(:,iter)>0.5,iter) = 1;
    lambda(iter) = sum(W(:,iter))/T;
    % Maximization Step
    ols = feval('ols',DEP,[INDS.*W(:,iter) INDS.*(1-W(:,iter)) IND]);
    beta(:,iter) = ols.beta;
    sigma(iter) = ols.sige;
    r2(iter) = ols.rsqr;
    crit = corr( W(:,iter) , W(:,iter-1) );
    if iter == maxit;
        warning('em: maximum number of iterations exceeded.');
        break
    end
end
%---------------------------------------------------

% Function outputs
%---------------------------------------------------
% Conditional probability of collusion
output.W = W(:,iter);
% Prediction of the regime
output.Shat = Shat(:,iter);
% Unconditional probability of collusion
output.lambda = lambda(:,iter);
% Estimated coefficients
output.beta = beta(:,iter);
% Estimated standard deviation of the error
output.sigma = sigma(iter);
% R-squared
output.r2 = r2(iter);
% Historical estimates obtained until convergence
output.HW = W(:,1:iter);
output.HShat = Shat(:,1:iter);
output.Hlambda = lambda(:,1:iter);
output.Hbeta = beta(:,1:iter);
output.Hsigma = sigma(1:iter);
output.Hr2 = r2(1:iter);
```

```
% Start value for the conditional probability of collusion
output.crit0 = crit0;
%--------------------------------------------------
```

Appendix J

Matlab code for the new OLS EM algorithm

```
function output = em1(DEP,INDS,IND)

% New OLS EM Algorithm
%---------------------------------------------------
% output = em1(DEP,INDS,IND,S)
% DEP = Dependent variable vector
% INDS = Switching independent variables vector
% IND = Non-switching independent variables vector
%---------------------------------------------------

% Example
%---------------------------------------------------
% Supply equation: P = CMG + c(1)*S*Q + C(2)*(1-S)*Q + U2,
%         where S is the unobserved state variable
%
% Command to estimate the supply equation as a switching regression:
% output = em1(P,Q,CMG)
%---------------------------------------------------

% Maximum number of iterations
maxit = 100;

% Dimensions
T = size(DEP,1);
k = 2*size(INDS,2)+size(IND,2);

% Initial Values
%---------------------------------------------------
% Note: OLS function is available in Appendix L
ols=feval('ols',DEP,[INDS IND]);
if mean(DEP-[INDS IND]*ols.beta)<median(DEP-[INDS IND]*ols.beta);
    crit0 = 1-normcdf(ones(T,1)*prctile(DEP-[INDS IND]*...
        ols.beta,25),DEP-[INDS IND]*ols.beta,sqrt(ols.sige));
else
```

```
    crit0 = 1-normcdf(ones(T,1)*prctile(DEP-[INDS IND]*...
        ols.beta,75),DEP-[INDS IND]*ols.beta,sqrt(ols.sige));
end
%---------------------------------------------------

% Creation of variables
%---------------------------------------------------
W = zeros(T,maxit);
Shat = zeros(T,maxit);
lambda = zeros(1,maxit);
beta = zeros(k,maxit);
sigma = zeros(1,maxit);
r2 = zeros(1,maxit);
%---------------------------------------------------

% First iteration
%---------------------------------------------------
iter = 1;
% Expectation step
W(:,1) = crit0;
Shat(W(:,1)>0.5,1) = 1;
lambda(1) = sum(W(:,1))/T;
% Maximization step
ols=feval('ols',DEP,[INDS.*Shat(:,1) INDS.*(1-Shat(:,1)) IND]);
beta(:,1) = ols.beta;
sigma(1) = ols.sige;
r2(1) = ols.rsqr;
%---------------------------------------------------

% Second and remaining iterations
%---------------------------------------------------
crit=0;
while  crit < 0.99999;
    iter = iter+1;
    % Expectation step
    h0 = normpdf(DEP,[zeros(T,1) INDS IND]*beta(:,iter-1),...
        sqrt(sigma(1,iter-1)));
    h1 = normpdf(DEP,[INDS zeros(T,1) IND]*beta(:,iter-1),...
        sqrt(sigma(1,iter-1)));
```

```
    W(:,iter) = lambda(iter-1)*h1 ./ ( lambda(iter-1)*h1 +...
        (1-lambda(iter-1))*h0 );
    Shat(W(:,iter)>0.5,iter) = 1;
    lambda(iter) = sum(W(:,iter))/T;
    % Maximization Step
    ols = feval('ols',DEP,[INDS.*Shat(:,iter)...
        INDS.*(1-Shat(:,iter)) IND]);
    beta(:,iter) = ols.beta;
    sigma(iter) = ols.sige;
    r2(iter) = ols.rsqr;
    crit = corr( W(:,iter) , W(:,iter-1) );
    if iter == maxit;
        warning('em: maximum number of iterations exceeded.');
        break
    end
end
%---------------------------------------------------


% Function outputs
%---------------------------------------------------
% Conditional probability of collusion
output.W = W(:,iter);
% Prediction of the regime
output.Shat = Shat(:,iter);
% Unconditional probability of collusion
output.lambda = lambda(:,iter);
% Estimated coefficients
output.beta = beta(:,iter);
% Estimated standard deviation of the error
output.sigma = sigma(iter);
% R-squared
output.r2 = r2(iter);
% Historical estimates obtained until convergence
output.HW = W(:,1:iter);
output.HShat = Shat(:,1:iter);
output.Hlambda = lambda(:,1:iter);
output.Hbeta = beta(:,1:iter);
output.Hsigma = sigma(1:iter);
output.Hr2 = r2(1:iter);
```

```
% Start value for the conditional probability of collusion
output.crit0 = crit0;
%---------------------------------------------------
```

Appendix K

Matlab code for the TSLS EM algorithm

```
function output = em2(DEP,ENDS,EXO,IV)

% TSLS EM Algorithm
%---------------------------------------------------
% output = em2(DEP,ENDS,EXO,IV,S)
% DEP = Dependent variable vector
% ENDS = Switching endogenous variables vector
% EXO = Exogenous variables vector
% IV = Instrumental variables
%---------------------------------------------------

% Example
%---------------------------------------------------
% System of equations:
% Demand: Q = c(1) + c(2)*Y + c(3)*P + U1
% Supply: P = CMG + c(4)*S*Q + C(5)*(1-S)*Q + U2,
%         where S is the unobserved state variable
%
% Command to estimate the supply as a switching regression:
% output = em2(P,Q,CMG,[C Y])
%---------------------------------------------------

% Maximum number of iterations
maxit = 200;

% Dimensions
T = size(DEP,1);
k = 2*size(ENDS,2);
g = size(EXO,2);

% Initial Values
%---------------------------------------------------
% Note: OLS function is available in Appendix L
ols1 = feval('ols',ENDS,[EXO IV]);
ols2 = feval('ols',DEP,[ols1.yhat EXO]);
```

```
beta0 = ols2.beta;
DEPHAT0 = [ENDS EXO]*beta0;
sigma0 = (DEP-DEPHAT0)'*(DEP-DEPHAT0)/(T-k-g);
if mean(DEP-[ENDS EXO]*beta0)<median(DEP-[ENDS EXO]*beta0);
    crit0 = 1-normcdf(ones(T,1)*prctile(DEP-[ENDS EXO]*...
        beta0,25),DEP-[ENDS EXO]*beta0,sqrt(sigma0));
else
    crit0 = 1-normcdf(ones(T,1)*prctile(DEP-[ENDS EXO]*...
        beta0,75),DEP-[ENDS EXO]*beta0,sqrt(sigma0));
end
%---------------------------------------------------

% Creation of variables
%---------------------------------------------------
W = zeros(T,maxit);
Shat = zeros(T,maxit);
lambda = zeros(1,maxit);
beta = zeros(k+g,maxit);
ENDSHAT0 = zeros(T,maxit);
ENDSHAT1 = zeros(T,maxit);
DEPHAT = zeros(T,maxit);
DEPHAT0 = zeros(T,maxit);
DEPHAT1 = zeros(T,maxit);
sigma = zeros(1,maxit);
r2 = zeros(1,maxit);
sigma0 = zeros(1,maxit);
sigma1 = zeros(1,maxit);
%---------------------------------------------------

% First iteration
%---------------------------------------------------
iter=1;
% Expectation step
W(:,1) = crit0;
Shat(W(:,1)>0.5,1)=1;
lambda(1) = sum(W(:,1))/T;
% Maximization step
ols1 = feval('ols',ENDS,[EXO IV repmat(Shat(:,1),1,...
    size(EXO,2)).*EXO repmat(Shat(:,1),1,size(IV,2)).*IV]);
```

```
ols2 = feval('ols',DEP,[ols1.yhat.*Shat(:,1)...
    ols1.yhat.*(1-Shat(:,1)) EXO]);
beta(:,1) = ols2.beta;
DEPHAT(:,1) = [ENDS.*Shat(:,1) ENDS.*(1-Shat(:,1)) EXO]*beta(:,1);
sigma(1) = (DEP-DEPHAT(:,1))'*(DEP-DEPHAT(:,1))/(T-k-g);
r2(1) = 1 - ( (DEP-DEPHAT(:,1))'*(DEP-DEPHAT(:,1)) ) /...
    ( (DEP-mean(DEP))'*(DEP-mean(DEP)) );
ENDSHAT0(:,1) = [EXO IV zeros(T,size(EXO,2)).*EXO...
    zeros(T,size(IV,2)).*IV] * ols1.beta;
ENDSHAT1(:,1) = [EXO IV ones(T,size(EXO,2)).*EXO...
    ones(T,size(IV,2)).*IV] * ols1.beta;
DEPHAT0(:,1) = [zeros(T,1) ENDSHAT0(:,1) EXO]*beta(:,1);
DEPHAT1(:,1) = [ENDSHAT1(:,1) zeros(T,1) EXO]*beta(:,1);
sigma0(1) = (DEP.*(1-Shat(:,1))-DEPHAT0(:,1).*(1-Shat(:,1)))'*...
    (DEP.*(1-Shat(:,1))-DEPHAT0(:,1).*(1-Shat(:,1)))/...
    (T-sum(Shat(:,1))-k/2-g);
sigma1(1) = (DEP.*Shat(:,1)-DEPHAT1(:,1).*Shat(:,1))'*...
    (DEP.*Shat(:,1)-DEPHAT1(:,1).*Shat(:,1))/(sum(Shat(:,1))-k/2-g);
%---------------------------------------------------

% Second and remaining iterations
%---------------------------------------------------
crit=0;
while  crit < 0.99999;
    iter = iter+1;
    % Expectation step
    h0 = normpdf(DEP,[zeros(T,1) ENDSHAT0(:,iter-1) EXO]*...
        beta(:,iter-1),sqrt(sigma0(iter-1)));
    h1 = normpdf(DEP,[ENDSHAT1(:,iter-1) zeros(T,1) EXO]*...
        beta(:,iter-1),sqrt(sigma1(iter-1)));
    W(:,iter) = lambda(iter-1)*h1 ./ ( lambda(iter-1)*h1 +...
        (1-lambda(iter-1))*h0 );
    Shat(W(:,iter)>0.5,iter) = 1;
    lambda(iter) = sum(W(:,iter))/T;
    % Maximization Step
    ols1 = feval('ols',ENDS,[EXO IV repmat(Shat(:,iter),1,...
        size(EXO,2)).*EXO repmat(Shat(:,iter),1,size(IV,2)).*IV]);
    ols2 = feval('ols',DEP,[ols1.yhat.*Shat(:,iter) ols1.yhat.*...
        (1-Shat(:,iter)) EXO]);
```

```matlab
        beta(:,iter) = ols2.beta;
        DEPHAT(:,iter) = [ENDS.*Shat(:,iter)...
            ENDS.*(1-Shat(:,iter)) EXO]*beta(:,iter);
        sigma(iter) = (DEP-DEPHAT(:,iter))'*...
            (DEP-DEPHAT(:,iter))/(T-k-g);
        r2(iter) = 1-( (DEP-DEPHAT(:,iter))'*(DEP-DEPHAT(:,iter)) ) /...
            ( (DEP-mean(DEP))'*(DEP-mean(DEP)) );
        ENDSHAT0(:,iter) = [EXO IV zeros(T,size(EXO,2)).*EXO...
            zeros(T,size(IV,2)).*IV] * ols1.beta;
        ENDSHAT1(:,iter) = [EXO IV ones(T,size(EXO,2)).*EXO...
            ones(T,size(IV,2)).*IV] * ols1.beta;
        DEPHAT0(:,iter) = [zeros(T,1) ENDSHAT0(:,iter) EXO]*beta(:,iter);
        DEPHAT1(:,iter) = [ENDSHAT1(:,iter) zeros(T,1) EXO]*beta(:,iter);
        sigma0(iter) = (DEP.*(1-Shat(:,iter))-DEPHAT0(:,iter).*...
            (1-Shat(:,iter)))'*(DEP.*(1-Shat(:,iter))-DEPHAT0(:,iter)...
            .*(1-Shat(:,iter)))/(T-sum(Shat(:,iter))-k/2-g);
        sigma1(iter) = (DEP.*Shat(:,iter)-DEPHAT1(:,iter).*...
            Shat(:,iter))'*(DEP.*Shat(:,iter)-DEPHAT1(:,iter).*...
            Shat(:,iter))/(sum(Shat(:,iter))-k/2-g);
        crit = corr( W(:,iter) , W(:,iter-1) );
        if iter == maxit;
            warning('em: maximum number of iterations exceeded.');
            break
        end
    end
    %---------------------------------------------------

    % Function outputs
    %---------------------------------------------------
    % Conditional probability of collusion
    output.W=W(:,iter);
    % Prediction of the regime
    output.Shat=Shat(:,iter);
    % Unconditional probability of collusion
    output.lambda=lambda(:,iter);
    % Estimated coefficients
    output.beta=beta(:,iter);
    % Estimated standard deviation of the error
    output.sigma=sigma(iter);
```

```
% R-squared
output.r2 = r2(iter);
% Historical estimates obtained until convergence
output.HW=W(:,1:iter);
output.HShat=Shat(:,1:iter);
output.Hlambda=lambda(:,1:iter);
output.Hbeta=beta(:,1:iter);
output.Hsigma=sigma(:,1:iter);
output.Hr2 = r2(1:iter);
% Start value for the conditional probability of collusion
output.crit0=crit0;
%---------------------------------------------------
```

Appendix L

Matlab code for the OLS estimator

```
function output=ols(y,x)

% OLS Estimator
%---------------------------------------------------
% output = ols(y,x)
% y = Dependent variable vector
% x =  independent variables vector
%---------------------------------------------------

% Errors
%---------------------------------------------------
if (nargin ~= 2); error('Wrong # of arguments to ols');
else
 [nobs,nvar] = size(x); [nobs2,~] = size(y);
 if (nobs ~= nobs2); error('x and y must have same # obs in ols');
 end;
end;
%---------------------------------------------------

% Matrix inv(x'x)
%---------------------------------------------------
if nobs < 10000
  [~,r] = qr(x,0);
  xpxi = (r'*r)\eye(nvar);
else
  xpxi = (x'*x)\eye(nvar);
end;
%---------------------------------------------------

% Function output
%---------------------------------------------------
output.meth = 'ols';
% Dependent variable
output.y = y;
```

```
% Number of observations
output.nobs = nobs;
% Number of variables
output.nvar = nvar;
% Estimated coefficients
output.beta = xpxi*(x'*y);
% Fitted dependent variable
output.yhat = x*output.beta;
% Residual
output.resid = y - output.yhat;
sigu = output.resid'*output.resid;
% Estimated variance of the error
output.sige = sigu/(nobs-nvar);
tmp = (output.sige)*(diag(xpxi));
sigb=sqrt(tmp);
% Standard deviation of the estimators
output.bstd = sigb;
tcrit=-tdis_inv(.025,nobs);
% 5% confidence intervals
output.bint=[output.beta-tcrit.*sigb, output.beta+tcrit.*sigb];
% T-statistics
output.tstat = output.beta./(sqrt(tmp));
ym = y - mean(y);
rsqr1 = sigu;
rsqr2 = ym'*ym;
% R-squared
output.rsqr = 1.0 - rsqr1/rsqr2;
rsqr1 = rsqr1/(nobs-nvar);
rsqr2 = rsqr2/(nobs-1.0);
if rsqr2 ~= 0
% Rbar-squared
output.rbar = 1 - (rsqr1/rsqr2);
else
    output.rbar = output.rsqr;
end;
ediff = output.resid(2:nobs) - output.resid(1:nobs-1);
% Durbin-watson
output.dw = (ediff'*ediff)/sigu;
%---------------------------------------------------
```

References

Abrantes-Metz, R. M., Froeb, L. M., Geweke, J. F., & Taylor, C. T. (2006). A variance screen for collusion. *International Journal of Industrial Organization*, 24(3), 467-486.

Abreu, D. (1984). Infinitely Repeated Games with Discounting: A General Theory. *Harvard Institute of Economic Research, Harvard University: Discussion Paper*, 1083.

Almoguera, P. A., Douglas, C. C., & Herrera, A. M. (2011). Testing for the cartel in OPEC: non-cooperative collusion or just non-cooperative? *Oxford Review of Economic Policy*, 27(1), 144 -168.

Bain, J. S. (1951). Relation of profit rate to industry concentration: american manufacturing, 1936-1940. *The Quarterly Journal of Economics*, 65(3), 293-324.

Bajari, P., & Ye, L. (2003). Deciding between competition and collusion. *The Review of Economics and Statistics*, 85(4), 971-989.

Baldwin, L. H., Marshall, R. C., & Richard, J. F. (1997). Bidder collusion at forest service . *The Journal of Political Economy*, 105(4), 657-699.

Bergès, F., & Caprice, S. (2008). Is Competition or Collusion in the Product Market Relevant for Labour Markets? *Recherches économiques de Louvain*, 74(3), 273-298.

Bertrand, J. (1883). Book review of theorie mathematique de la richesse sociale and of recherches sur les principles mathematiques de la theorie des richesses. *Journal de Savants*, 67, 499-508.

Bhaskar, V., Manning, A., & To, T. (2002). Oligopsony and Monopsonistic Competition in Labor Markets. *Journal of Economic Perspectives*, 16(2), 155-174.

Bloch, K. (1932). On German cartels. *The Journal of Business of the University of Chicago*, 5(3), 213-222.

Bond, Schoeneck & King, LLP. (February de 2002). Salary Survey May Violate Antitrust Law. *Business Law Information Memo* .

Bresnahan, T. F. (1989). Empirical studies of industries with market power. In R. Schmalensee, & R. Willig, *Handbook of Industrial Organization, Volume 2*. Amsterdam: Elsevier.

Brod, A., & Shivakumar, R. (1999). Advantageous semi-collusion. *The Journal of Industrial Economics*, 47(2), 221-230.

Chen, J., & Kalbfleisch, J. D. (2005). Modified likelihood ratio test in finite mixture models with a structural parameter. *Journal of Statistical Planning and Inference*, 129, 93-107.

Chow, G. C. (1960). Tests of Equality Between Sets of Coefficients in Two Linear Regressions. *Econometrica*, 28(3), 591-605.

Christin, C. (2009). Collusive Strategic Buying of a Necessary Input. *Work in Progress*.

Cournot, A. (1838). *Researches into the Mathematical Principles of the Theory of Wealth.* The Macmillan Company.

Dannemann, J., & Holzmann, H. (2010). Testing for two components in a switching regression model. *Computational Statistics & Data Analysis*, 54(6), 1592-1604.

d'Aspremont, C., & Jacquemin, A. (1988). Cooperative and noncooperative R&D in duopoly with spillovers. *American Economic Review*, 78(5), 1133-1137.

Deltas, G., & Serfes, K. (2002). Semicollusion vs. full collusion: the role of demand uncertainty and product substitutability. *Journal of Economics*, 77(2), 111-139.

Deltas, G., Serfes, K., & Sicotte, R. A. (1999). American shipping cartels in the pre-World War I era. *Research in Economic History*, 19, 1-38.

Dowd, J. M. (1996). Oligopsony Power: Antitrust Injury and Collusive Buyer Practices in Input Markets. *Boston University Law Review*, 76, 1076-1116.

Eckard, E. W. (1991). Competition and the cigarette TV advertising ban. *Economic Inquiry*, 29(1), 119-133.

Fershtman, C., & Muller, E. (1986). Capital investments and price agreements in semicollusive markets. *Rand Journal of Economics*, 17(2), 214-226.

Friedman, J. W. (1971). A Non-Cooperative Equilibrium for Supergames. *Review of Economic Studies*, 38(113), 1-12.

Goldfeld, S. M., & Quandt, R. E. (1972). *Nonlinear methods in econometris.* Amsterdam: North-Holland Publishing Co.

González, A., & Ayala, L. (2012). Does input purchase cooperation foster downstream collusion? *Serie de Documentos de Trabajo del Departamento de Economia de la Universidad de Chile*, SDT 358.

Green, E. J., & Porter, R. H. (1984). Noncooperative Collusion under Imperfect Price Information. *Econometrica*, 52(1), 87-100.

Hamilton, J. L. (1994). Joint oligopsony-oligopoly in the U.S. leaf tobacco market, 1924-39. *Review of Industrial Organization*, 9(1), 25-39.

Hammond, S. D. (2000, November 22). Detecting and Deterring Cartel Activity Through an Effective Leniency Program. *Department of Justice*.

Harrington Jr., J. E. (2005). Detecting cartels. *Advances in the Economics of Competition Law*.

Hipple, S. F. (2010, September). Self-employment in the United States. *Monthly Labor Review*, pp. 17-32.

Hotelling, H. (1929). Stability in Competition. *Economic Journal*, 39(153), 41-57.

Kiefer, N. M. (1980). A note on switching regressions and logistic discrimination. *Econometrica*, 48(4), 1065-1069.

Kiefer, N. M. (1978). Discrete parameter variation: efficient estimation of a switching regression model. *Econometrica*, 46(2), 427-434.

Lorange, P. (1973). Anatomy of a complex merger: a case study and analysis. *Journal of Business Finance*, 5, 32-38.

Manning, A. (2010). Imperfect Competition in the Labour Market. *CEP Discussion Paper No 981*.

Manning, A. (2003). The Real Thin Theory: Monopsony in Modern Labour Markets. *Labour Economics*, 10(2), 105-131.

Morgenstern, O. (1976). The Collaboration Between Oskar Morgenstern and John von Neumann on the Theory of Games. *Journal of Economic Literature*, 14(3), 805-816.

Mukherjee, A., Selvaggi, M., & Vasconcelos, L. (2012). Star Wars: Exclusive Talent and Collusive Outcomes in Labor Markets. *Journal of Law Economics & Organization* , 28(4), 754-782.

Nasar, S. (1998). *A Beautiful Mind: The Life of Mathematical Genius and Nobel Laureate John Nash*. New York: Simon and Schuster.

Nash, J. F. (1950). The Bargaining Problem. *Econometrica*, 18(2), 155-162.

Nash, J. F. (1953). Two-Person Cooperative Games. *Econometrica*, 21(1), 128-140.

Nash, J. (1951). Non-Cooperative Games. *Annals of Mathematics*, 54(2), 286-295.

Nevo, A. (2001). Measuring market power in the ready-to-eat cereal industry. *Econometrica*, 69(2), 307-342.

Paha, J. (2011). Empirical methods in the analysis of collusion. *Empirica*, 38(3), 389-415.

Peters, M. (2010). Labor Markets After the Black Death: Landlord Collusion and the Imposition of Serfdom in Eastern Europe and the Middle East. *Prepared for the Stanford Comparative Politics Workshop*.

Porter, R. H. (1983). A study of cartel stability: the joint executive committee, 1880-1886. *The Bell Journal of Economics*, 14(2), 301-314.

Porter, R., & Zona, D. (1993). Detection of bid-rigging in procurement auctions. *Journal of Political Economy*, 101(3), 518-538.

Poundstone, W. (1992). *Prisioner's Dilema*. New York: Doubleday.

Quandt, R. E. (1972). A new approach to estimating switching regressions. *Journal of the American Statistical Association*, 67 (338), 306-310.

Quandt, R. E. (1960). Tests of the hypothesis that a linear regression system obeys two separate regimes. *Journal of the American Statistical Association*, 55(290), 324-330.

Salop, S. C. (1979). Monopolistic Competition with Outside Goods. *The Bell Journal of Economics*, 10(1), 141-156.

Selten, R. (1973). A Simple Model of Imperfect Competition, where 4 are Few and 6 are Many. *International Journal of Game Theory*, 2(1), 141-201.

Shelkova, N. Y. (2008). Low-Wage Labor Markets and the Power of Suggestion. *Economics Working Papers*, Paper 200833.

Simbanegavi, W. (2009). Informative advertising: competition or cooperation? *Journal of Industrial Economics*, 57(1), 147-166.

Slade, M. E. (2004). Market power and joint dominance in U.K. brewing. *The Journal of Industrial Economics*, 52(1), 133-163.

Smith, A. (1776). *An Inquiry into the Nature and Causes of the Wealth of Nations.* London: W. Strahan and T. Cadell.

Steen, F., & Sørgard, L. (2009). Semicollusion. *Foudations and Trends in Microeconomics*, 5(3), 153-228.

Stigler, G. J. (1964). A Theory of Oligopoly. *Journal of Political Economy*, 72, 44-61.

von Neumann, J., & Morgenstern, O. (1943). *Theory of Games and Economic Behavior*. New Jersey: Princeton University Press.

Zhu, H.-T., & Zhang, H. (2004). Hypothesis testing in mixture regression models. *Journal of the Royal Statistical Society: Series B*, 66(1), 3-16.